STRESSES IN A SOIL MASS

A stress on the soil depends on the load per unit area. Construction of a foundation mainly increases the stresses on the soil.

It is necessary to estimate the net increase of vertical stresses acting upon the soil as a result of construction of a foundation so that we can calculate the settlement strategy.

As the stress increases in the soil, the soil can be deformed by the stress.

There are two stresses can acts on a plane: normal stress (σ) and shear stress(τ) .

Stresses acting on a soil plane:

Consider a two dimensional soil element where all the stresses are acting on the side of the soil element.

The normal stress is always perpendicular to the shear stress. Here we are going to determine the normal stress and the shear stress acting on a plane which makes some angle θ to the side of that block.

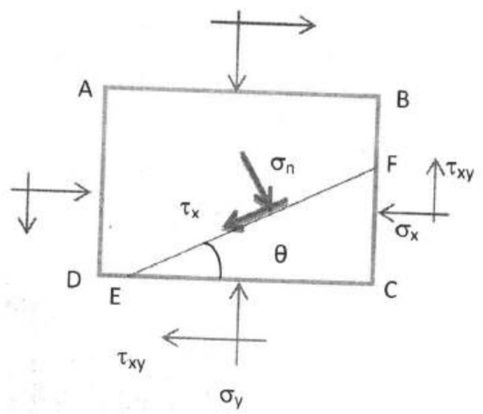


Fig.1 ABCD is a soil block which has been subjected into the stresses of normal and shear stress.

To determine stresses on the plane EF, we should draw a free body diagram of the triangle EFC

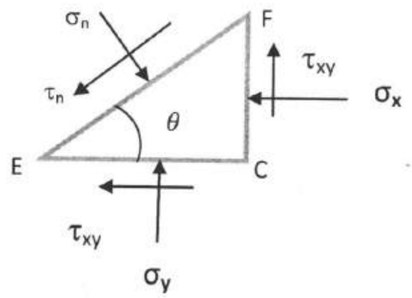


Fig.2 Free body diagram of EFC as shown in (Fig.1)

Since these are equal and opposite forces are acting on the block they are cancelling to each other , so this block is in a steady state.

Let σ_n and τ_n are the normal stress and shear stress respectively, on the plane EF.

From geometry, we know that

$$EC = EF\cos\theta$$

$$FC = EF\sin\theta$$

Total normal force acting on EF plane is F1

F1 = σ_n EF, where EF is the area and σ_n is the stress .

$$EF\,\sigma_n = FC\,\sigma_x \sin\theta + FC\,\tau_{xy}\cos\theta + \sigma_y EC\cos\theta + \tau_{xy} FC\sin\theta$$

$$EF\,\sigma_n = EF\;\sigma_x \sin^2\theta + \tau_{xy}\,EF\sin\theta\cos\theta + EF\;\sigma_y\cos^2\theta + EF\;\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_n = \sigma_x \sin^2\theta + 2\,\tau_{xy}\sin\theta\cos\theta + \sigma_y\cos^2\theta$$

$$\boldsymbol{\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_y - \sigma_x}{2}\cos 2\theta + \tau_{xy}\sin 2\theta}$$

Normal stress acting on plane EF.

Again , $\tau_n \text{EF} = \sigma_x \text{ EF } \sin\theta\cos\theta + \sigma_y \text{EF } \sin\theta\cos\theta - \tau_{xy} \text{EF } \cos^2\theta + \tau_{xy} \text{EF } \sin^2\theta$

So, $\tau_n = \sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta - \tau_{xy} (\cos^2\theta - \sin^2\theta$

$$\boldsymbol{\tau_n = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta}$$

Shear stress acting on the plane EF

From this equation we can choose the value of θ in such a way that τ_n will be equal to zero.

Lets put $\tau_n = 0$,

$$\frac{\sigma_y - \sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta = 0$$

$$\tan 2\theta = \tau_{xy} \Big/ \frac{\sigma_y - \sigma_x}{2}$$

$$2\theta = \tan^{-1}\left(\tau_{xy} \Big/ \frac{\sigma_y - \sigma_x}{2}\right)$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\tau_{xy} \Big/ \frac{\sigma_x + \sigma_y}{2}\right)$$

The plane that makes this angle with the block, which only has normal stress, called as principal plane.

There are two principal stresses can formed by making an angle 90^0 to each other where shear stress is zero.

Major principal stress = $\boldsymbol{\sigma_n = \sigma_1 = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_y-\sigma_x}{2}\right)^2 + \tau_{xy}^2}}$

Minor principal stress = $\boldsymbol{\sigma_n = \sigma_3 = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_y-\sigma_x}{2}\right)^2 + \tau_{xy}^2}}$

Normal stress and shear stress acting on any plane can be determined by the plotting Mohr's circle also, where each and every point on the circumference of circle is a stress component.

To plot the stresses on the Mohr's circle, we should use the sign convection of the normal and shear stress. Normal stresses are taken as positive in all cases and shear stress to be positive if the stress produces the motion of an element in a counter clockwise direction.

TO PLOT STRESSES ON THE SOIL ELEMENT IN A MOHR'S CIRCLE,

Co-ordinates :

Plane AD,

Normal stress is $+\sigma_x$ and shear stress is $+\tau_{xy}$

Plane CD,

Normal stress is $+\sigma_x$ and shear stress is $-\tau_{xy}$ (it produces clockwise rotation)

Centre of the Mohr's circle : $\frac{\sigma_x + \sigma_y}{2}$

Where $\sigma_x = \sigma_1\,(maximum\ stress),\ \sigma_y = \sigma_3(minimum\ stress)$

Radius of the circle : $R^2 = (x-a)^2 + y^2$

$$= \left(\sigma_n - \frac{\sigma_x+\sigma_y}{2}\right)^2 + \tau_n^2$$

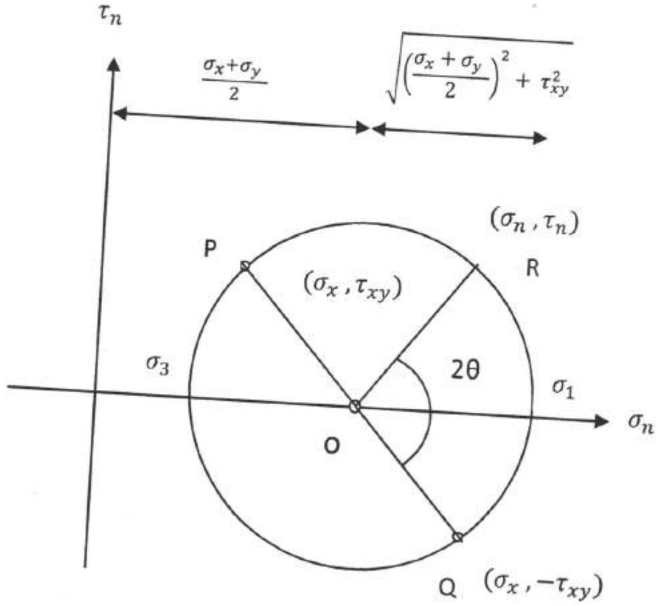
$$= \left(\frac{\sigma_y-\sigma_x}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right)^2 + \left(\frac{\sigma_y-\sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta\right)^2$$

$$= \left(\frac{\sigma_y-\sigma_x}{2}\right)^2 + \tau_{xy}^2$$

$$R = \sqrt{\left(\frac{\sigma_y-\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

Here P and Q represent the stress conditions on the plane AD and CD of the soil element.

O is the point of intersection of normal stress axis with line PQ.



The stress on the plane EF can be determined by moving by an angle 2θ , which is twice of the angle made by the plane EF with plane CD, in a counter clockwise direction from point Q along the circumference of the circle and reach to the point R. The co-ordinate of the point R gives normal stress and shear stress of plane EF.

STRESS DISTRIBUTION

Stress induced in a soil mass is due to over lying soil over that particular soil mass and the over burden load over it.

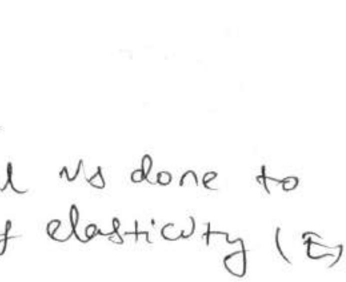
- Before construction need following things need to analyze

a. Stability analysis of soil
b. Settlement analysis of the soil (Due to load applied on it)
c. Earth pressure (active or passive pressure)

* As the soil is not homogeneous hence the analysis of stress strain behaviour of soil is very complex process.

The stress strain behaviour depends on :-
- drainage condition of soil
- void ratio of soil
- water content
- applied load
- Stress path

So to make the analysis easier assuming that the soil is homogeneous material & it is isotropic, the stress strain behaviour is linear it will obey the theory of elasticity



Stress ↑ strength ↑

$$E = \frac{\text{stress}}{\text{strain}}$$

Stress Strain Parameter of Soil
- soil is a non elastic material
- The stress strain analysis of soil is done to determine the value of modulous of elasticity (E) & poisson's ratio (μ)

E ⇒ For gravel the value of stress is high as compare to clay

Stress $\propto$ E

Hence For gravel E(↑) as compare to clay, sand, silt

- The poussion's ratio will vary from 0 to 0.5 for elastic body.
- As soil is non elastic so the value of poussion's ratio will not lie within this range.
- For clay the value of poussion's ratio is more as compare to gravel.

UU Test is conducted over saturated cohesive soil

$\nu = 0.5$

- CD test is carried out for cohesionless soil

$\nu = < 0.5$

The modulus of elasticity of soil can also determine by triaxial test.

- The secant modulus is always half or 1/3 peak stress value

$(\sigma_1 - \sigma_3)$ is the axial stress applied on the soil sample

ε_1 = is the axial strain / axial deformation in the soil mass.

Geostatic Stresses :-

The stress develop by the self wt. of soil mass is much higher as compare to the stress develop by the applied load over the soil mass.

- But in case of steel structure stress is generated due to the external load & very small portion of stress is generated due to its self wt.
- When the ground surface is horizontal & the properties of soil donot change along the horizontal direction the stress generated to the self wt. of soil mass is known as geostatic stress.

Ex. This type of condition is exist in case of sedimentary soil.

- The stresses are normal to horizontal & vertical plane on soil mass and there is no shearing force over that plane
- The plane on which the normal stress is acting is called principal plane

Symbols to be Used for Stress Distribution:

The total stress field at point within a soil mass loaded at its boundary consists of nine stress components given below:

$$\begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

These nine stress components as given by this group of square matrix of stress are the components of a mathematical entry called the 'stress tensor' of a symmetrical matrix relative to its main diagonal. The main diagonal elements (upper left (or) lower right) of a stress tensor are the normal stress components, and the off diagonal elements are shear stress out of the nine stress components indicated above, there are three independent shear components making the total unknowns to be equal to six. The corresponding nine strain components are given by the following strain tensor.

$$\begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix}$$

Where ε denotes the linear or direct strain & γ denotes the shearing strain

Stresses Due to the Self weight of soil :-

Consider the stresses within a soil mass due to its own weight stresses due to self weight are some times known as geo static stress. Let us take the soil mass to be bounded by the horizontal plane (ground surface) xy & z axis be directed downwards. Under this condition, the soil mass is said to be semi infinite where there is no external loading, the ground plane becomes or principal plane since it is devoid of any shear loading. From the symmetry and orthogonally plane since it is devoid of any shear loading. From the symmetry and orthogonally of principal planes, are can conclude that all the horizontal and vertical planes will be devoid of shear stress so that within soil mass.

$\tau_{xy} = \tau_{xz} = \tau_{yz} = 0$ substituting this in equilibrium eqn we get $\sigma_z = \gamma z$ — ①

where γ = unit weight of soil

σ_z = vertical stress at a point within and soil mass, situated at a depth z below ground surface

Similarly from compatibility eqns in terms of stresses (for 3 dimensional case)

$$\sigma_x = \sigma_y = \frac{\mu}{1-\mu}\gamma z = k_o \gamma z \quad — ②$$

μ = Poisson's ratio

$k_o = \frac{\mu}{1-\mu}$ k_o = Coefficient of pressure at rest

Thus the eqn ① & ② give stress components at a point situated at a depth z below the ground surface, due to self wt. of soil mass above it.

At a certain point within the soil mass, the stress components due to both these loading. (Self wt. & surface loading)

Vertical stress :-

- Consider a horizontal section at a depth z below the ground surface the cross sectional area of the horizontal element is consider as A

γ = unit wt. of soil

vertical stress $\sigma_z = \gamma z$

Boussinesq's Theory / Equation :-

Boussinesq (1885) solved the problem of stress distribution in soils due to a concentrated load acting at a ground surface by assuming a suitable stress function.

Assumptions :-

1. The soil medium is an elastic, homogeneous, isotropic and semi infinite (half space)
2. Soil obeys Hooke's law
3. Self weight of soil is ignored
4. Soil is initially unstressed
5. Change in volume upon the application of load is neglected.
6. Top surface is free of shear stress
7. Continuity of stress exists in the medium
8. Stresses are distributed symmetrically w.r.t. Z axis

Homogeneous means at different locations, soil has same elastic properties in same direction (Same E, μ)

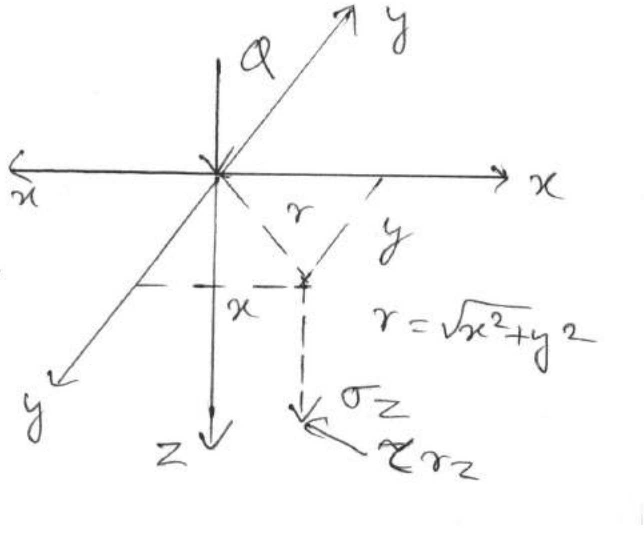
Isotropic means at a single point, soil has same elastic properties in different directions.

Semi infinite means material bounded by a horizontal plan and extending to infinite length in all directions to one side of horizontal plane

Vertical stress

$$\sigma_z = \frac{Q}{z^2}\,\frac{3}{2\pi}\left[\frac{1}{1+(r/z)^2}\right]^{5/2}$$

$$\sigma_z = \frac{Q}{z^2} K_B$$

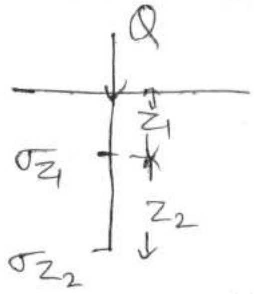


k_B = Boussinesq's Influence factor

If $r = 0$ (vertically below the load)

$$\sigma_z = \frac{Q}{z^2} \frac{3}{2\pi}$$

$$\Rightarrow \sigma_z \propto \frac{1}{z^2}$$

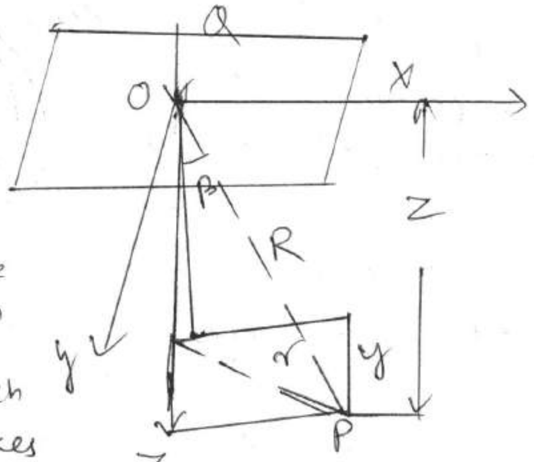


$$\frac{\sigma_{z_1}}{\sigma_{z_2}} = \left(\frac{z_2}{z_1}\right)^2$$

Radial shear stress $\tau_{rz} = \sigma_z \frac{r}{z}$

vertically below the load $\tau_{rz} = 0$

OR



O - origin point

Q = point load acting on the origin

R = Polar distance between the origin & point P

β = angle at which the line OP makes an angle with the vertical plane

$$R = \sqrt{x^2 + y^2 + z^2}$$

σ_R = Polar stress generated at point P

$$\sigma_z = K_B \times \frac{Q}{z^2}$$

K_B = Boussinesq's Coefficient of vertical stress

$$K_B = \frac{3}{2\pi} \left[\frac{1}{1 + (r/z)^2}\right]^{5/2}$$

z = vertical distance of the point below the load

r = it is the radial distance of the point

$$r = \sqrt{x^2 + y^2}$$

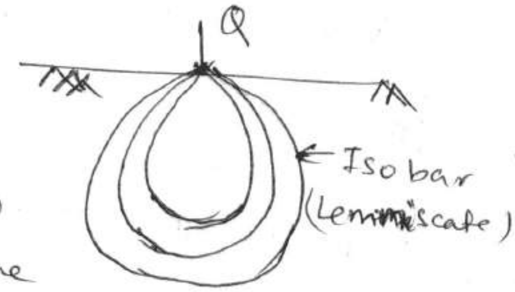
radial shear stress $\tau_{rz} = \sigma_z (r/z)$

Pressure distribution diagram :-

By means of Boussinesq's stress distribution theory the following vertical pressure distribution diagrams can be prepared.

1. Stress isobar or isobar diagram
2. vertical pressure distribution on a horizontal plane
3. vertical pressure distribution on a vertical line.

Isobar :- It is a curve or contour connecting all points below the ground surface of equal vertical stress.

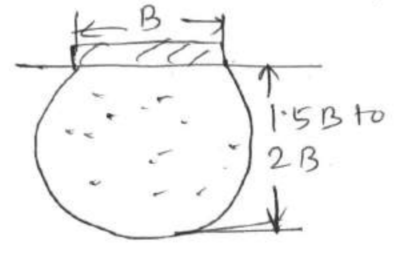


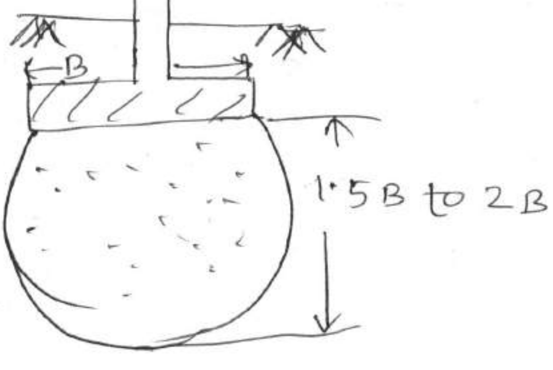
An isobar is a spatial, curved surface of the shape of a bulb because the vertical pressure on a given horizontal plane is the same in all directions at points located at equal radial distances around the axis of loading

Pressure bulb :-

It is the zone of the soil in which there is significant stress. Beyond the pressure bulb, stress in the soil is negligible.

⇒ The zone in a loaded soil mass bounded by an isobar of given vertical pr. intensity is called pressure bulb. In the case of footings, the depth of the pressure bulb is taken as 1.5 B to 2B (as shown in fig)





Suppose an isobar of value

$\sigma_z = 0.25\,Q$ or 25% of Q per unit area is to be plotted

From eqn.

$$K_B = \frac{\sigma_z \times z^2}{Q} = \frac{0.25\,Q\,z^2}{Q} = 0.25\,z^2$$

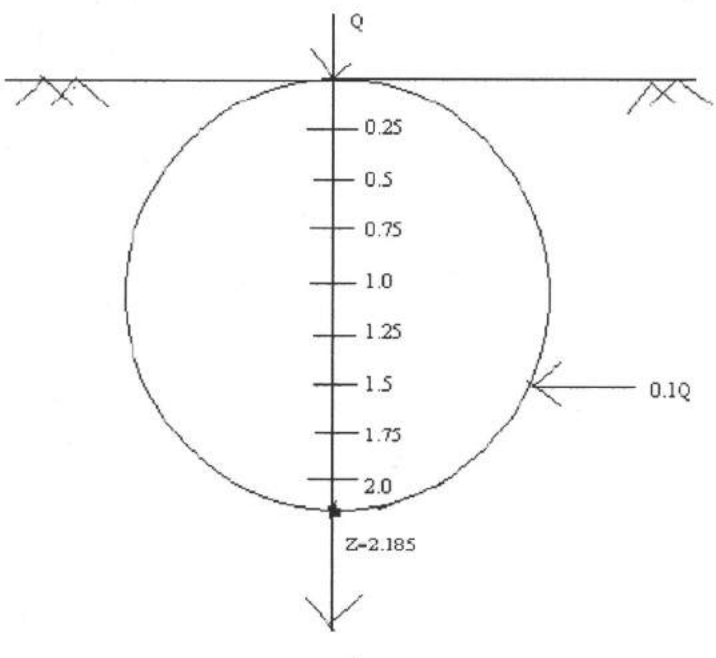
Isobar: An isobar is a curve joining the points of equal stress intensity. An isobar is a spatial curved surface of shape of an electric bulb. The curved surface is symmetrical about the vertical axis passing through the point. Isobar of intensity 0.1 Q/ unit area

$\sigma_Z = K_B (Q / Z^2)$

$0.1 Q = K_B (Q / Z^2)$

$K_B = 0.1 Z^2$

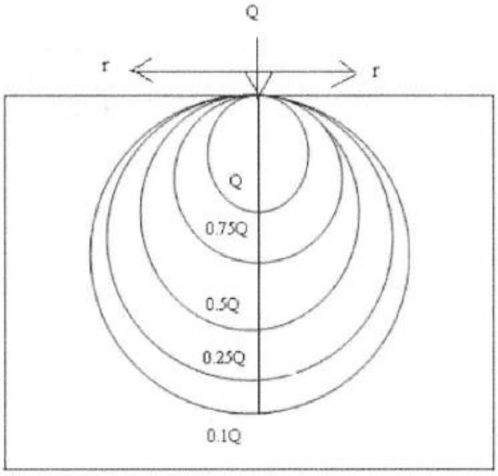
Similarly isobars of value $K_B = 0.25 Z^2$ Different intensity of loading such as 0.2Q, 0.3Q etc. Isobars are useful for determining the effect of load on the vertical stresses at various points. The zone within which the stresses have a significant effect on the settlement of stress id known as pressure bulb. It is generally assumed as isobar of 0.1Q form pressure bulb. The area outside the pressure bulb is assumed to have negligible.



$K_B = 0.25 Z^2$

A number numerical values of Z is selected and thus values of K_B are calculated. Corresponding to this value of K_B, r/Z are formed and hence corresponding values of 'r' are computed. When r = 0, $K_B = 0.4775$

Z units	KB	r/Z	r
0.2	0.0100	1.92	0.38
0.4	0.0400	1.30	0.52
0.6	0.0900	0.97	0.58
0.8	0.1600	0.74	0.59
1.0	0.2500	0.54	0.54
1.2	0.3600	0.34	0.41
1.38	0.4775	0.00	0.00



Vertical stress distribution on a horizontal plane:

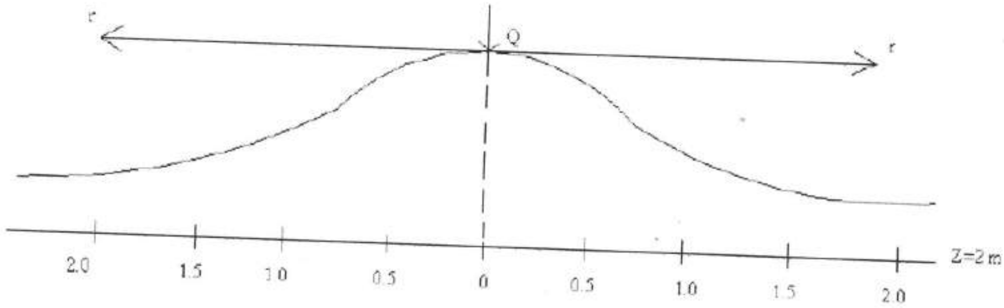
The vertical stress at various points on a horizontal plane at a particular depth Z can be obtained using $\sigma_Z = K\,(Q / Z^2)$

Let us consider Z = 2 m, then $\sigma_Z = K_B\,(Q / Z^2) = Q/Z^2 = Q/4$

The values of σ_Z are computed for different values of r/Z after obtaining I_B value.

$K_B = 3/ 2\pi \times 1 / [1+(r/Z)^2]^{5/2}$

r/z	K_B	σ_z
0.0	0.4775	0.1194Q
0.25	0.4103	0.1026Q
0.5	0.2733	0.0683Q
0.75	0.1565	0.0390Q
0.1	0.0844	0.0211Q



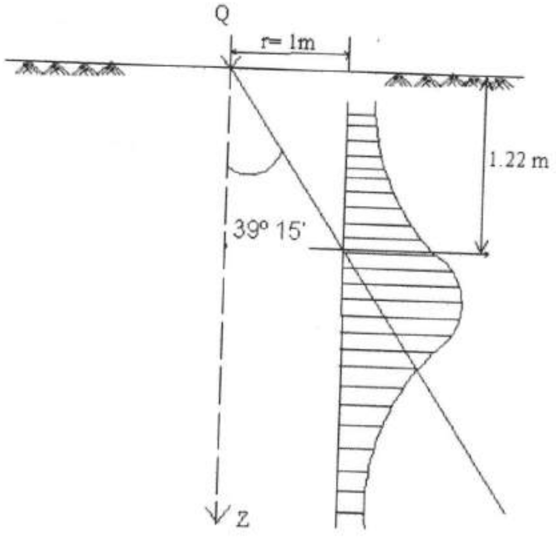
Vertical stress distribution on a horizontal plane:

In this case radial distance r is constant as depth changes. The value of r/Z obtained for different values of $\sigma_Z = K_B (Q / Z^2)$

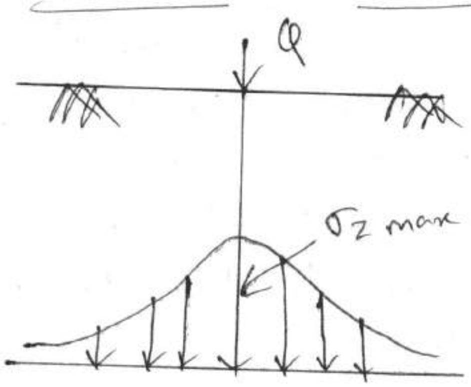
Let us consider r = 1m,

r/Z	K_B	σ_z
4.0	0.004	0.0064Q
2.0	0.0085	0.0340Q
1.0	0.0844	0.0844Q
0.667	0.1094	0.0845Q
0.5	0.2733	0.0687Q
0.4	0.3294	0.0527Q
0.2	0.4329	0.017Q

It is noted that vertical stress first increases and then decreases. The maximum vertical stress occur at r/Z = 0.817. This corresponds to the point of intersection of the vertical plane with the line drawn at 39°15′ to the vertical axis of the load.



Vertical Stress distribution on Horizontal plane :-

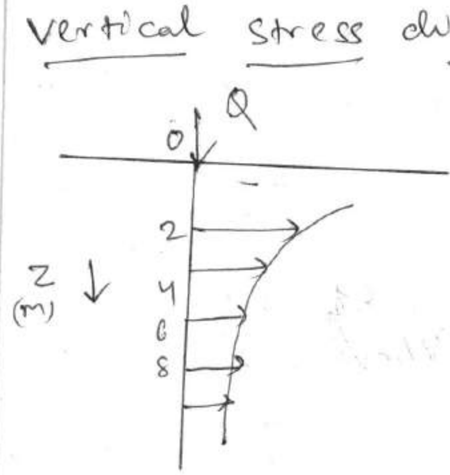


The vertical pressure distribution on any horizontal plane at depth z below ground surface due to a Concentrated load is given by

$$\sigma_z = K_B \frac{Q}{z^2}$$

Depth z is known depth selecting different values of horizontal distance r

Vertical stress distribution on a Vertical line :-



σ_z also decreases with increase in the depth z on any vertical line distance r from the axis of the load, the variation of σ_z Can be plotted from the relation σ_z

$$\sigma_z = K_B \, Q/z^2$$

The above expression the radial distance 'r' associated with K_B is constant. Hence various values of z & r/z can be selected & K_B can be found. Then σ_z can be computed which will be proportional of K_B/z^2.

Vertical pressure due to a line load :-

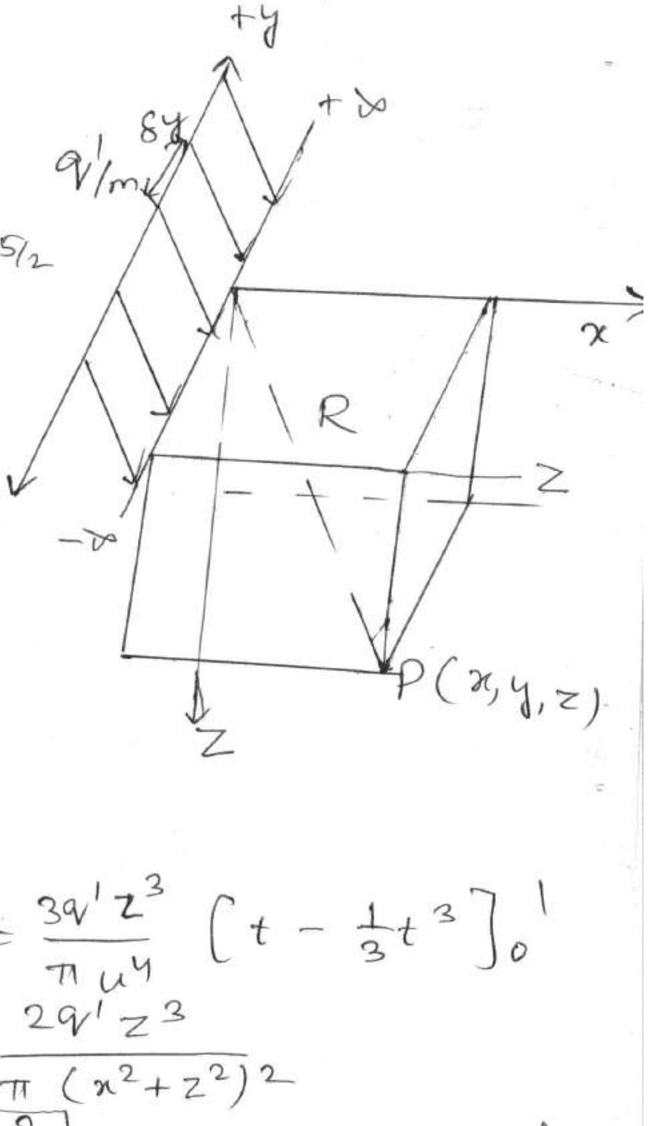
If the line load is of intensity q' per unit length ∥ll to y-axis on the surface of semi infinite elastic medium. Let us consider the load acting on a small length 'δy'. The load can be taken as a point load of $q'\delta y$ & Boussinesq's solⁿ can be applied to determine vertical stress at point $P(x, y, z)$

$$\Delta\sigma_z = \frac{3q'\delta y}{2\pi} = \frac{z^3}{(r^2+z^2)^{5/2}} \quad \text{— (1)}$$

The vertical stress at P due to line load extending from $-\infty$ to $+\infty$ obtained by integration.

$$\sigma_z = \frac{3q' z^3}{2\pi}\int_{-\infty}^{+\infty}\frac{dy}{(r^2+z^2)^{5/2}}$$

$$= \frac{3q' z^3}{2\pi}\int_{-\infty}^{+\infty}\frac{dy}{(x^2+y^2+z^2)^{5/2}}$$



Substituting

$$x^2+z^2 = u^2$$

$$\sigma_z = \frac{3q' z^3}{2\pi}\int_{-\infty}^{+\infty}\frac{dy}{(u^2+y^2)^{5/2}}$$

Let $y = u\tan\theta$

$dy = u\sec^2\theta \, d\theta$

$$\sigma_z = \frac{3q' z^3}{2\pi}\int_0^{\pi/2}\frac{u\sec^2\theta}{u^5\sec^5\theta}\,d\theta$$

$$\sigma_z = \frac{3q' z^3}{2\pi u^4}\int_0^{\pi/2}\cos^3\theta \, d\theta$$

Let $\sin\theta = t$ $\quad \cos\theta \, d\theta = dt$

$$\sigma_z = \frac{3q' z^3}{\pi u^4}\int_0^1 (1-t^2)\,dt = \frac{3q' z^3}{\pi u^4}\left[t - \frac{1}{3}t^3\right]_0^1$$

$$= \frac{3q' z^3}{\pi u^4} \times \frac{2}{3} = \frac{2q' z^3}{\pi (x^2+z^2)^2}$$

$$\boxed{\sigma_z = \frac{2q'}{\pi z}\left[\frac{1}{1+(x/z)^2}\right]^2}$$

Point P vertically below the line $x = 0$

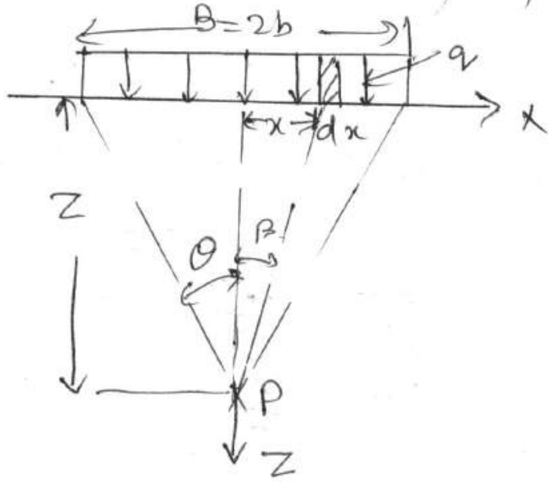
$$\sigma_z = \frac{2q'}{\pi z}$$

$$\tau_{xz} = \frac{2q'}{\pi}\left[\frac{xz^2}{(x^2+z^2)^2}\right]$$

Vertical Stress under a strip load:-

Case I
(i) Point P below the centre of the strip:-

Let A strip load of width $B = 2b$ and intensity q. Let us consider the load acting on a small elementary width dx at a distance x from the centre of the load. Small load of $q\,dx$ can be considered as a line load of intensity q'



$$\Delta\sigma_z = \frac{2q\,dx}{\pi z}\left[\frac{1}{1+(x/z)^2}\right]^2$$

The vertical stress due to entire strip load

$$\sigma_z = \frac{2q}{\pi z}\int_{-b}^{+b}\frac{dx}{\left[1+(x/z)^2\right]^2}$$

$x/z = \tan u \quad dx = z\sec^2 u\,du$

$$\sigma_z = \frac{2q}{\pi z}\times 2\int_0^{\theta}\frac{z\sec^2 u\,du}{(1+\tan^2 u)^2}\,du$$

where $\theta = \tan^{-1}(b/z)$ = angle made by extremities of the strip at P

$$\sigma_z = \frac{4q}{\pi}\int_0^{\theta}\cos^2 u\,du$$

$$\sigma_z = \frac{4q}{\pi}\int_0^{\theta}\left(\frac{1+\cos 2u}{2}\right)du$$

$$\boxed{\sigma_z = \frac{q}{\pi}(2\theta + \sin 2\theta)}$$

(ii) Point P not below the centre of the strip:-

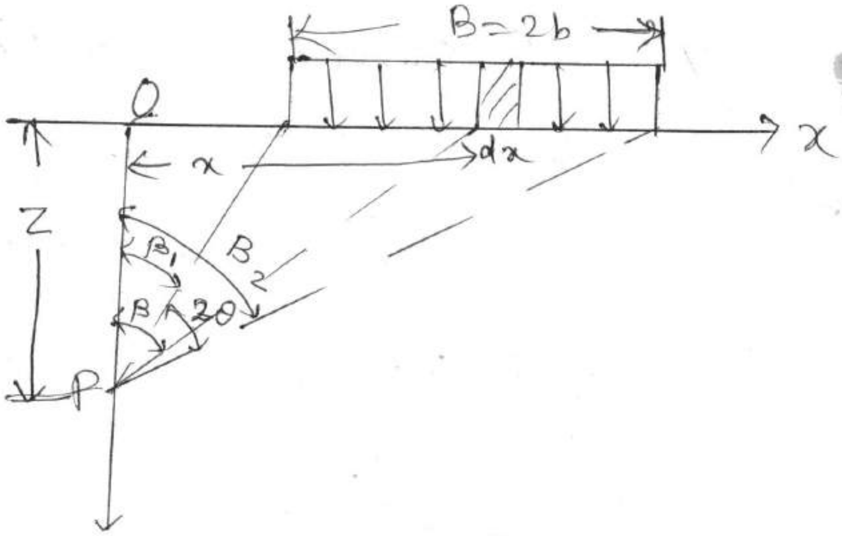
Point P not below the centre of the strip. When the point P is not below the centre of the strip make angles of β_1 & β_2. The extremities of the strip make a small length at P. The load $q\,dx$ acting on a small length dx can be considered as a line load

The vertical stress at P is given by

$$\Delta\sigma_z = \frac{2q\,dx}{\pi z}\left[\frac{1}{1+(x/z)^2}\right]^2$$

$x = z \tan \beta$

or $dx = z \sec^2 \beta \, d\beta$



$$\Delta \sigma_z = \frac{2q(z \sec^2\beta)\, d\beta}{\pi z} \left[\frac{1}{1+\tan^2\beta} \right]^z$$

$$\Delta \sigma_z = \frac{2q}{\pi} \cos^2\beta \, d\beta$$

$$\sigma_z = \frac{q}{\pi} \int_{\beta_1}^{\beta_2} (1+\cos 2\beta)\, d\beta$$

Integrate $= \frac{q}{\pi} \left[\beta + \frac{\sin 2\beta}{2} \right]_{\beta_1}^{\beta_2}$

$$\sigma_z = \frac{q}{\pi} \left[(\beta_2 - \beta_1) + (\sin \beta_2 \cos\beta_2 - \sin\beta_1 \cos\beta_1) \right]$$

$\beta_2 - \beta_1 = 2\theta$

$$\sigma_z = \frac{q}{\pi} \left[2\theta + (\sin \beta_2 \cos\beta_2 - \sin\beta_1 \cos\beta_1) \right]$$

$(\sin \beta_2 \cos\beta_2 - \sin\beta_1 \cos\beta_1) \rightarrow \sin 2\theta \cos 2\phi$

If $\beta_1 + \beta_2 = 2\phi$

$$\sigma_z = \frac{q}{\pi} \left[2\theta + \sin 2\theta \cos 2\phi \right]$$

$$\sigma_x = \frac{q}{\pi} \left[2\theta - \sin 2\theta \cos 2\phi \right]$$

$$\tau_{xz} = \frac{q}{\pi} \left[\sin 2\theta \sin 2\phi \right]$$

When P is below the centre of load

In this case $\beta_2 = -\beta_1 = \theta$ & $\beta_1 + \beta_2 = 0$ or $\phi = 0$

$$\sigma_z = \frac{q}{\pi} (2\theta + \sin 2\theta)$$

Shear stress at any point P below a strip load is given by

$$\boxed{\tau_{xz} = \frac{q}{\pi} \sin 2\theta \sin 2\phi}$$

Vertical stress under a circular area:-

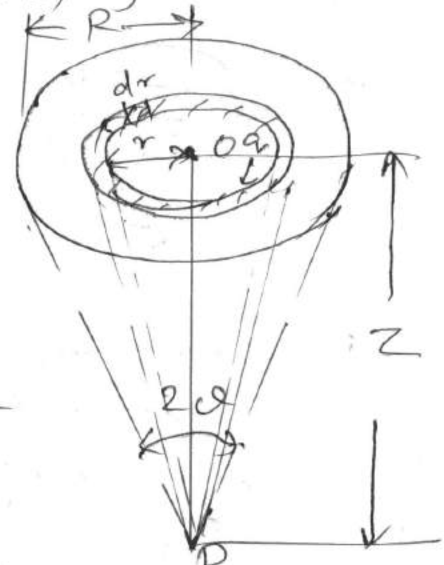
The loads applied to soil surface by footing are not concentrated loads. These are usually spread over a finite area of the footing. It is generally assumed that the footing is flexible & contact pressure is uniform. The load is assumed to be uniformly distributed over the area of the base of footings.

Let the intensity of the load be q per unit area & R be the radius of the loaded area. Boussinesq's soln can be used to determine σ_z

The load on the elementary ring of radius r & width dr is equal to $q(2\pi r\,dr)$. The load acts at a constant radial distance r from point P.

$$\Delta\sigma_z = \frac{3(q \times 2\pi r\,dr)}{2\pi}\frac{1}{z^2}\left[\frac{1}{1+(r/z)^2}\right]^{5/2}$$

The vertical stress due to entire load



$$\sigma_z = 3qz^3\int_0^R \frac{r\,dr}{(r^2+z^2)^{5/2}}$$

Let $r^2+z^2 = u$ or $2r\,dr = du$

$$\sigma_z = 3qz^3\int_{z^2}^{R^2+z^2}\frac{du}{2u^{5/2}}$$

$$= \frac{3}{2}qz^3\left(\frac{-2}{3}\right)\left[u^{-3/2}\right]_{z^2}^{R^2+z^2}$$

$$= -qz^3\left[\frac{1}{(R^2+z^2)^{3/2}} - \frac{1}{(z^2)^{3/2}}\right]$$

$$= qz^3\left[\frac{1}{z^3} - \frac{1}{(R^2+z^2)^{3/2}}\right]$$

$$\boxed{\sigma_z = q\left[1 - \left\{\frac{1}{1+(R/z)^2}\right\}^{3/2}\right]}$$

$$\sigma_z = I_c\,q$$

I_c = Influence coefficient for circular area

Westergaard's solution:-

Boussinesq's soln assumes that the soil deposit is isotropic. Actual sedimentary deposits are generally anisotropic. There are generally thin layers of sand embedded in homogeneous clay strata. Westergaard's soln assumes that there are thin sheets of rigid materials sandwiched in a homogeneous soil mass. These thin sheets are closely packed and are of infinite rigidity and are incompressible. These permit only downward displacement of the soil mass as a whole without any lateral displacement.

According to Westergaard the vertical stress at a point P at a depth z below the concentrated load Q is

$$\sigma_z = \frac{Q}{z}\frac{1}{2\pi}\left[\frac{\sqrt{\frac{1-2\mu}{2(1-\mu)}}}{\left[\frac{1-2\mu}{2(1-\mu)}+(r/z)^2\right]}\right]^{3/2}$$

$$\eta = \sqrt{\frac{1-2\mu}{2(1-\mu)}}$$

$$\sigma_z = \frac{Q}{2\pi\eta^2 z^2}\left[\frac{1}{1+(r/\eta z)^2}\right]^{3/2}$$

For elastic material μ varies from 0 to 0.5

For a case of large lateral restraint the lateral strain is very small & $\mu = 0$ (assume)

$$\boxed{\sigma_z = \left[\frac{1}{1+2(r/z)^2}\right]^{3/2}\frac{Q}{\pi z^2}} = K_W\frac{Q}{z^2}$$

$$K_W = \frac{1}{\pi}\left[\frac{1}{1+2(r/z)^2}\right]^{3/2}$$

For circular load

$$\sigma_z = q\left\{1-\left[\frac{1}{1+(a/\eta z)^2}\right]^{1/2}\right\}$$

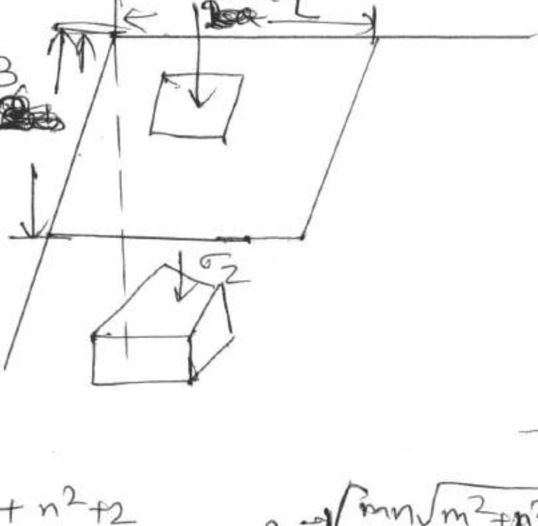
a = radius

q = load intensity

Vertical Stress under a corner of Rectangular area:-

The vertical stress under a corner of a rectangular area with a uniformly distributed load of intensity q can be obtained from Boussinesq's solution.

The stress at depth z is given by



$$dQ = q\,dA = q\,dx\,dy$$

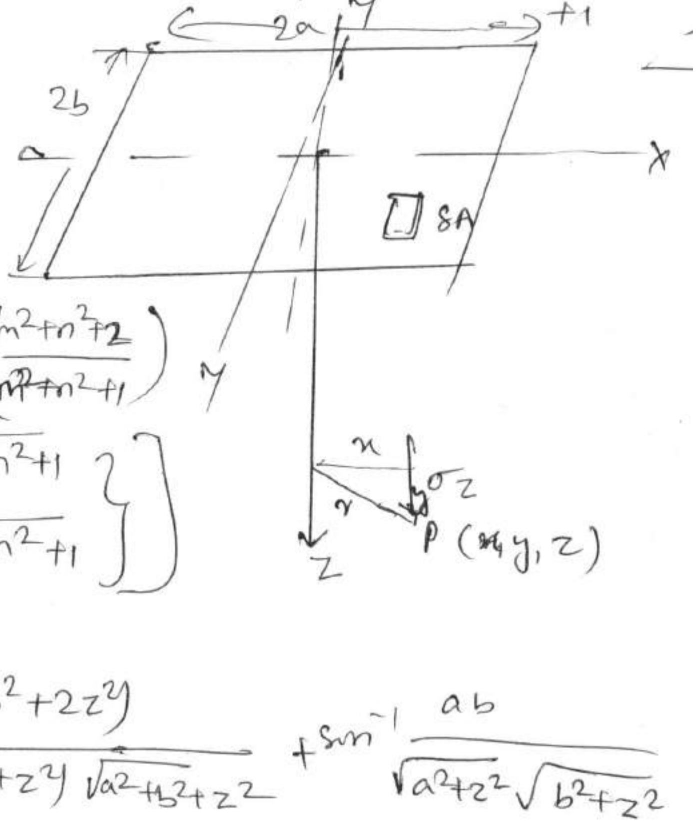
$$\Delta\sigma_z = \frac{3(q\,dx\,dy)z^3}{2\pi}\,\frac{1}{(x^2+y^2+z^2)^{5/2}}$$

By integration

$$\sigma_z = \frac{3q\,z^3}{2\pi}\int_0^L\int_0^B \frac{q\,dx\,dy}{(x^2+y^2+z^2)^{5/2}}$$

$$m = B/z \quad n = L/z$$

$$\sigma_z = \frac{q}{2\pi}\left[\frac{mn}{\sqrt{m^2+n^2+1}}\cdot\frac{m^2+n^2+2}{m^2+n^2+m^2n^2+1} + \sin^{-1}\left(\frac{mn\sqrt{m^2+n^2+1}}{\sqrt{m^2+n^2+m^2n^2+1}}\right)\right]$$



$$(\sigma_z)_{corner} = \frac{q}{4\pi}\left[\frac{2mn\sqrt{m^2+n^2+1}}{m^2+n^2+m^2n^2+1}\times\left(\frac{m^2+n^2+2}{m^2+n^2+1}\right) + \tan^{-1}\left\{\frac{2mn\sqrt{m^2+n^2+1}}{m^2+n^2-m^2n^2+1}\right\}\right]$$

$$m = a/z \;\&\; n = \frac{b}{z}$$

$$(\sigma_z)_0 \text{ (centre of rectangle)} = \frac{2q}{\pi}\left[\frac{abz(a^2+b^2+2z^2)}{(a^2+z^2)(b^2+z^2)\sqrt{a^2+b^2+z^2}} + \sin^{-1}\frac{ab}{\sqrt{a^2+z^2}\sqrt{b^2+z^2}}\right]$$

Newmark's influence chart:

A more accurate method of determining the vertical stress at any point under a uniformly loaded area of any shape is with the help of influence chart or influence diagram original suggested by Newmark (1942). A chart, consisting of number of circles and radiating lines, is so prepared that the influence of each area unit (formed in the shape of a sector between two concentric circles and two adjacent, radial lines) is the same at the centre of the circles, i.e., each area unit causes the equal vertical stress at the centre of the diagram.

Let a uniformly loaded circular area of radius r_1 cm be divided into 20 sectors (area units) as shown in fig. . If q is the intensity of loading, and σ_z is the vertical pressure at a depth Z below the centre of the area, each unit such as OA, B, exerts a pressure equal to $\frac{\sigma_z}{z\sigma}$ at the centre.

Vertical Pressure under a uniformly loaded circular Area

$$\sigma_z = q\left[1 - \frac{1}{\left(1+\frac{a}{z}\right)^2}\right]^{3/2}$$

Hence, from the above equation

$$\frac{\sigma_z}{20} = \frac{q}{20}\left[1 - \left\{\frac{1}{1+\left(\frac{r_1}{Z}\right)^2}\right\}^{3/2}\right] = i_f q$$

where i_f = influence value

$$= \frac{1}{20}\left[\left\{1 - \frac{1}{1+\left(\frac{r_1}{Z}\right)^2}\right\}^{3/2}\right]$$

If ir be made equal to an arbitrarily fixed value say 0.005.

We have

$$\frac{q}{20}\left[1 - \left\{\frac{1}{1-\left(\frac{r_1}{Z}\right)^2}\right\}^{3/2}\right] = 0.005q$$

Selecting the value of Z = 5 cm (say), the value of r1 solved from equation 13.30 comes out be 1.35 cm. Hence if a circle is drawn with radius r_1 = 1.35 cm and divided into 20 equal area units, each area unit will exert a pressure equal to 0.005 q intensity at a depth of 5cm. Let the radious of second concentric circle be equal to r_2 cm. By extending the twenty radial lines, the space between the two concentric circles is again divided into 20 equal area units; A_1 A_2 B_2 B_1 is one such area unit. The vertical pressure at the centre, due to each of these area nits is to be intensity 0.005 q. Therefore, the total pressure due to area units OA, B1 and $A_1A_2B_2B_1$ at depth z = 5 cm below the centre is 2 x 0.005 q. Hence from equation Vertical pressure due to OA_2 B_2

$$= \frac{q}{20}\left[1 - \left\{\frac{1}{1+\left(\frac{r_2}{Z}\right)^2}\right\}^{3/2}\right] = 2\times 0.005q$$

Substituting z = 5 cm, we get r_2 = 2.00 cm from the above relation. Similarly, the radii of 3^{rd}, 4^{th}, 5^{th}, 6^{th}, 7^{th}, 8^{th}, 9^{th} circles can be calculated as tabulated in table . The radius of 10_{th} circle is given b the following governing equation:

$$\frac{q}{20}\left[1 - \left\{\frac{1}{1+\left(\frac{r_{10}}{Z}\right)^2}\right\}^{3/2}\right] = 10\times 0.005q = \frac{q}{20}$$

From the above r_{10} = infinity.

Fig shows the influence chart drawn on the basis of table

To use the chart for determining the vertical stress at any point under the loaded area, the plan of the loaded area is first drawn on a tracing paper to such a scale that the length ABH (= 5 cm) drawn on the chart represents the depth to the point at which pressure is required. For example, if the pressure is to be found at a depth of 5m, the scale of plan will be 5cm = 5m , or 1cm = 1m. The plane of the loaded area is then 30 placed over the chart that the point below which pressure is required coincides with the centre of the chart. The point below which pressure is required may lie within or outside the loaded area. The total number of area units (including the fractions covered by the plan of the loaded area is counted. The vertical pressure is then calculated from the relation)

$$\sigma_A = 0.005q \times N_A$$

where, N_A = number of area units under the loaded area.

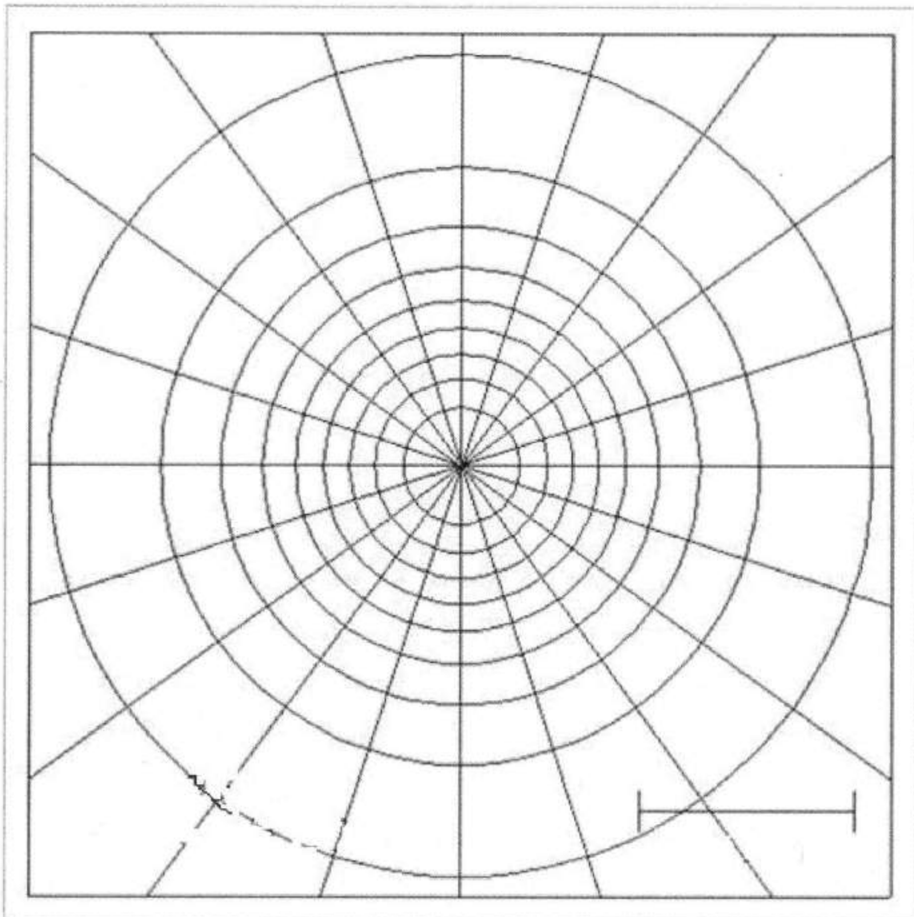


Figure: **Newmark's chart**

Application of Newmark's Influence chart

Application of Newmark' s Influence chart in solving problems is quite easy and simple. The plan of the loaded area is first drawn on a tracing sheet to the same scale as the scale of the line segment AB on the chart representing the depth 'z' . The location of the point where the vertical stress is required is marked on the plan, say as 'P' . Now, the tracing sheet is placed over the chart, such that the point 'P' comes exactly over the center of the chart from where the rays are emanating. Now the number of mesh covered by the plan is counted.

In case of partly covered mesh an intelligent judgement of the fraction of mesh covered is required. Let the total number of mesh be equal to 'n' . Then the vertical stress at the desired depth is given by:

sz = I x n x q

Where I = Influence value = 1/(c x s)

n = Number of meshes under the loaded area

q = uniformly distributed load

c = No. of concentric areas

s = No. of radial lines

Approximate method

The method discussed in the preceding sections are relatively more accurate, but are time consuming. Sometimes, the engineer is interested to estimate the vertical stresses approximately. For preliminary designs, thus saving time and labour without sacrificing accuracy to any significant degree.

They are also used to determine the stress distribution in soil under the influence of complex loading and/ or shapes of loaded areas.

Two commonly used approximate methods are:

Equivalent point- load method:-

- The vertical stress at a point under a loaded area of any of any shape can be determined by dividing the loaded area into small area and replacing the distributed load on each on small area by an equivalent point load acting at the centroid of the small area. The principle of superposition is then applied and the required stress at a specified point is obtained by summing up the contributions of the individual. Point loads from each of the units by applying the approximate point load formula, such as that of Boussinesq' s or Westergaard' s.

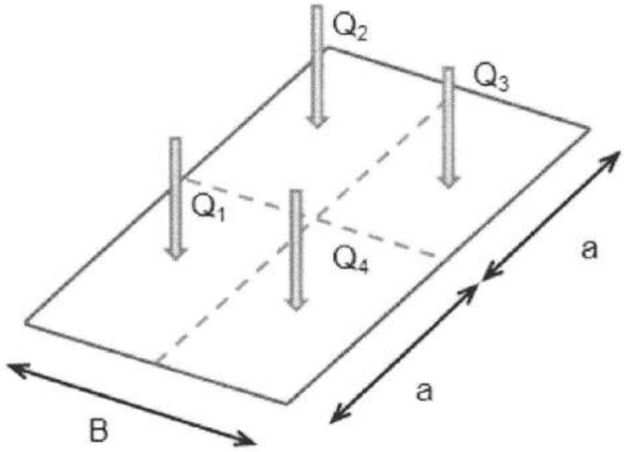


Figure: Equivalent point- load method

As shown in the above Figure, if a square area of size B is acted on by a uniform load q, the same area can be divided into four small area. And the load on each area can be converted into an equivalent point load assumed to act at its centroid. Then the vertical stress at any point below or outside the loaded area is equal to the sum of the vertical stresses due to these equivalent point loads. Then

$$\sigma_Z = \frac{[Q_1(I_B)_1 + Q_2(I_B)_2 + Q_3(I_B)_3 + \dots\dots\dots + Q_n(I_B)_n]}{Z^2} \quad \text{——— } 1$$

$$\sigma_Z = \frac{1}{Z^2} \sum_{i=1}^{n} Q_i (I_B)_i \quad \text{——— } 2$$

Note : Eq. 2 gives fairly accurate results if the side a of the small unit is equal to or less than one third of the depth Z at which the vertical stresses is required.

Two is to one (2:1) load distribution method:

The actual distribution of load with the depth is complex. However, it can be assumed to spread approximately at a slope of two (vertical) to one (horizontal). Thus the vertical pressure at any depth Z below the soil surface can be determined approximately by constructing a frustum of pyramid (or cone) of depth Z and side Slope(2:1), the pressure distribution is assumed to the uniform on a horizontal plane at that depth.

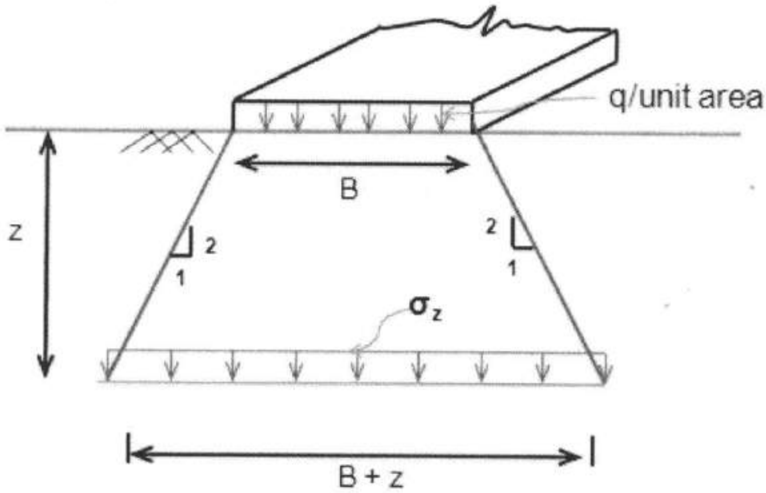


Fig. Two is to one (2:1) load distribution method

The average vertical stress σ_z ,depends upon the shape of the loaded area, as given below:

1) Square area (BxB) ; $\sigma_z = \dfrac{qB^2}{(B+z)^2}$

2) Rectangular area (B x L); $\sigma_z = \dfrac{q(B \times L)}{(B+z)(L+x)}$

3) Strip area (width B x unit length) $\sigma_z = \dfrac{q(B \times 1)}{(B+z)1}$

4) Circular area (diameter D) $\sigma_z = \dfrac{qD^2}{(D+z)^2}$

Contact pressure:

The upward pressure due to soil on the underside of the footing or foundation is termed contact pressure. In the derivations of vertical stress below the loaded areas using Boussinesq' s theory or Westergaard' s theory, it has been assumed that the footing is flexible and the contact pressure distribution is uniform and equal to 'q' . Actual footings are not flexible as assumed. The actual distribution of the contact pressure depends on a number of factors.

Factors affecting contact pressure distribution

The factors are:

1. Flexural rigidity of base of footing
2. Type of soil

3. Confinement

Flexural rigidity of base of footing

Uniform loading on a flexible base induces uniform contact pressure on any type of soil, while a rigid base induces non-uniform pressure. Foundation bases are usually thick massive concrete structures, which cannot be treated as ideally flexible.

Type of soil

The contact pressure distribution also depends on the elastic properties of the soil. The elastic properties of soil depends on the type of soil.

a) Flexible footing   b) Rigid footing

a. Sandy soil

Figure a & b shows the qualitative contact pressure distribution under flexible and rigid footing resting on a sandy soil and subjected to a uniformly distributed load q. when the footing is flexible, the edges undergo a large settlement than at centre. The soil at centre is confined and therefore has a high modulus of elasticity and deflects less for the same contact pressure. The contact pressure is uniform.

When the footing is rigid the settlement is uniform. The contact pressure is parabolic with zero intensity at the edge sand maximum at the centre.

b. Clayey soils

a) Flexible footing   ) Rigid footing

Fig. Contact pressure diagram on saturated clay

Fig. shows the qualitative contact pressure distribution under flexible and rigid footings resting on saturated clay and subjected to a uniformly distributed load q.

When the footing is flexible, it deforms into the shape of a bowel, with the maximum deflection at the centre. The contact pressure distribution is uniform. If the footing is rigid, the settlement is uniform. The contact pressure distribution is minimum at the centre and the maximum at the edges(infinite theoretically). The stresses at the edges in real soil cannot be infinite as theoretically determined for an elastic mass. In real soils, beyond a certain limiting values of stress, the plastic flow occurs and the pressure becomes infinite as shown in Fig.

c. C-Ø soil

For a c – Ø soil, the contact pressure for a flexible footing will be uniform as shown in Fig. (a). For a rigid footing, the pressure distribution will be as shown in the Fig. (b), it is more at the edge and less at the centre.

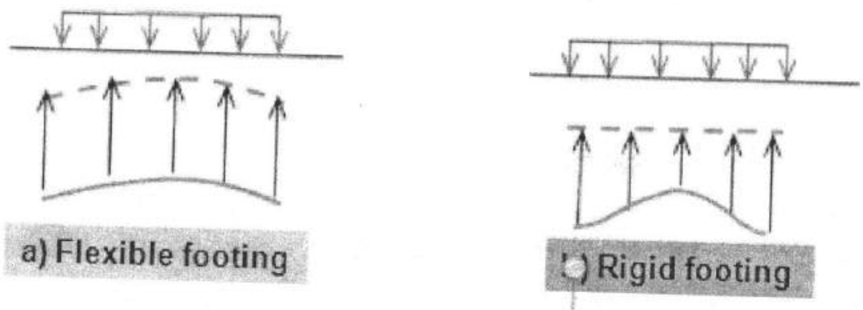


Fig. Contact pressure diagram on c – Ø soil

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