Lecture Notes

On

Advance Control Systems

7^{Th} Semester Electrical Engineering

Part 1: Modelling and Analysis of Control System using State Space Based Approach

By

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Course Objective:

The objective of this course is to equip students with a deep understanding of discrete-time control systems, state variable analysis, and nonlinear system behavior. Students will learn to analyze, design, and implement control strategies using modern techniques, including Z-transform methods, state-space representations, and Lyapunov stability analysis.

Course Outcome:

- **Analyze**: Analyze discrete-time and continuous-time control systems using Z-transform and state-space methods to determine system behavior.
- **Design**: Design control systems utilizing feedback strategies, pole placement, and observer design to achieve specified performance criteria.
- **Evaluate**: Evaluate the stability of linear and nonlinear systems using Routh's criterion and Lyapunov's methods, assessing their robustness.
- **Apply**: Apply techniques for modeling and simulating nonlinear systems, including phase plane and describing function methods, to solve practical engineering problems.

STATE SPACE ANALYSIS

Introduction:

The developement of control system analyses and design can be classified into three eras.

(a) Transfer Function based analysis and design techniques used before 1950's is called as <u>classical control Theory</u>.

Lack of adequate computing facilities make these methods quite popular as they relay on approximate graphical techniques to quickly arrive at design and analysis of the system.

These methods are more intitutive and provide rapidly transient and stability information.

(b) <u>Modern control theory</u> referes to the state space based methods developed in late 1950's.

Due to the developement in digital computers, it was possible to carry out analysis and design directly in time domain.

(c) The third era of control system developement started in the year of 1980's which combines the advantages of both classical and modern control theory. It is known as <u>robust control Theory</u>.

Motivation for State space Technique:

→ TF based frequency domain techniques are applicable to only limited range of systems. (Linear Time Invariant)

→ Analysis & design of MIMO system using TF based techniques are quite cumbersome.

→ TF based methods are not convinient for direct computer simulations as it is difficult to numerically compute laplace inversion and further partial fraction expansions are sensitive to root locations.

→ For space exploration requirement of control system designs increased to model and analyze time variant & non-linear systems as well.

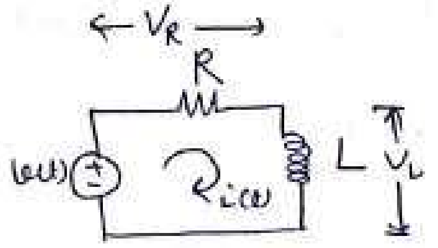
where as SS based approach is a direct time domain approach for modelling, analyzing and designing a wide range of systems (Linear / Non-linear, Time varying / Time invariant, SISO / MIMO)

Comparision between Frequency domain & Time domain Approach:

Frequency Domain Approach	Time Domain Approach
1. Requires systems with zero initial condition.	1. Can handle systems with non-zero initial conditions.
2. Restricted to limited range of systems. (LTI).	2. Applicable to wide variety of systems.
3. Cumbersome to handle MIMO system.	3. Can handle MIMO system with same level of complexity as SISO.
4. Not good for computer simulations.	4. Extremely good at computer simulations.
5. Extremely intitutive. i.e. with fewer calculation & using graphical techniques rapidly gives information about the transient & stability behaviour of the system.	5. Need elaborate computation to yield the physical interpretation of the model.
6. Provides greater depth of insight into feedback properties of the system.	6. Provides information on physical structure of the system.
7. It can handle plant uncertainty using gain & phase margins.	7. It requires accurate plant modelling for design and analysis.

State-space Modelling:

Let's consider a RL circuit.

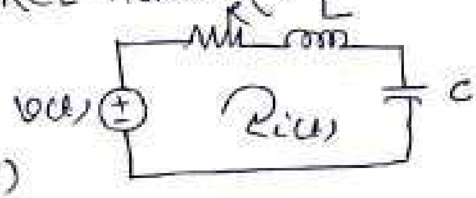


Different variables of this systems may be:–

v(t), $V_R(t)$, $V_L(t)$ & i(t)

Let's write the loop equation

Next, let's consider a 2nd order RCL network.



Writing loop equation,

$$v(t) = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \quad —(9)$$

Substituting $i = \frac{dq}{dt}$, eqn-(9) can be re-written as

$$v(t) = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{1}{C}q \quad —(10)$$

Since eqn-(10) is a 2nd order differential equation, it can be converted into two 1st order differential eqn in terms of $i(t)$ and $q(t)$.

i.e. $$\frac{dq}{dt} = i \quad —(11)$$

Substituting $\frac{dq}{dt} = i$ & solving for $\frac{di}{dt}$ in eqn-(10)

$$v(t) = Ri + L\frac{di}{dt} + \frac{1}{C}q$$

or, $$\frac{di}{dt} = -\frac{R}{L}i - \frac{1}{LC}q + \frac{1}{L}v(t) \quad —(12)$$

Here, $i(t)$ & $q(t)$ are k/as state variable
$v(t)$ is k/as input variable.
Eqn-(11) & (12) are k/as state equation

Let's compute other network variables,

$$v_R(t) = R\,i(t) \quad —(13)$$

$$v_C(t) = \frac{1}{C}q(t) \quad —(14)$$

$$v_L(t) = v(t) - R\,i(t) - \frac{1}{C}q(t) \quad —(15)$$

Again the remaining network variables are expressed algebraically in terms of state variable ($i(t)$ & $q(t)$) and i/p variable $v(t)$.

Hence eqn 11, 12, 13, 14 & 15 provides a viable representation of the network known as state space representation.

Equations (11) & (12) which describes the dynamics of the network is also not unique. To prove it let's take $v_R(t)$ & $v_C(t)$ to be new state variables.

$$v_R(t) = R\,i(t) \Rightarrow \frac{dv_R(t)}{dt} = R\frac{di(t)}{dt} = R\frac{1}{L}(v_L)$$

or, $$\dot{v}_R(t) = \frac{R}{L}\left[v(t) - v_R(t) - v_C(t)\right] \quad —(16)$$

$$v(t) = R\,i(t) + L\frac{di(t)}{dt} \quad \text{———(1)}$$

Taking Laplace Transform on above eqn,

$$V(s) = R\,I(s) + L[s\,I(s) - i(0)] \quad \text{———(2)}$$

Assuming input to be unit step, $v(t) = u(t) \Rightarrow V(s) = \frac{1}{s}$

$$\frac{1}{s} = R\,I(s) + Ls\,I(s) - L\,i(0)$$

solving for $I(s)$, we get

$$I(s) = \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right) + \frac{i(0)}{s+\frac{R}{L}} \quad \text{———(3)}$$

Taking ILT,

$$i(t) = \frac{1}{R}\left(1 - e^{-\frac{R}{L}t}\right) + i(0)\,e^{-\frac{R}{L}t} \quad \text{———(4)}$$

Observation:

(i) From above eqn it is concluded that, $i(t)$ for $t \geq 0$ can be computed if ~~v(t) and~~ $i(0)$ and $v(t)$ for $t \geq 0$ are known.

(ii) $V_R(t) = R\,i(t)$ ———(5)

$V_L(t) = v(t) - R\,i(t)$ ———(6)

$\frac{di}{dt} = \frac{1}{L}[v(t) - R\,i(t)]$ ———(7)

Other network variables such as $V_R(t)$, $V_L(t)$ & $\frac{di}{dt}$ can also be computed algebraically if $i(t)$ and $v(t)$ ~~are~~ known.

In state space approach, $i(t)$ is k/as state variable and $v(t)$ is k/as input variable.

Eqn-(1) is k/as state eqn & eqn-(5), (6) & (7) are k/as o/p eqn.

(iii) Equation (1) which describes the dynamics of the network is not unique. This equation could be written in terms of any other network variable.

substituting $i(t) = \frac{V_R}{R}$, in eqn-(1)

$$v(t) = V_R + \frac{L}{R}\frac{dV_R}{dt} \quad \text{———(8)}$$

Eqn-(8) can be solved by knowing $v(t)$ & $V_R(0) = R\,(i(0))$.

Here $V_R(t)$ will be k/as state variable. Other network variables can also be computed algebraically from $V_R(t)$ & $v(t)$.

Similarly, $V_c(t) = \frac{1}{C}\int i\,dt$

$$\frac{dV_c(t)}{dt} = \frac{1}{C} i = \frac{1}{C}\frac{V_R}{R} = \frac{1}{RC} V_R(t) \quad \text{—(17)}$$

Eqn -(16) & (17) are new state equations which also represents the dynamics of above network.

Hence, state space representation consists of
(i) First order differential eqn : To obtain state variables
(ii) Algebraic o/p eqn : To obtain rest of system variables

Restriction on choice of state variable:

(i) Minimum no. of state variables required to describe the system equals (the no. of energy storing element) order of differential equation.

(ii) Minimal set of state variables must be <u>linearly independent</u>.
In the previous example, $V_R(t)$ & $i(t)$ can not be choosen as state variables as both of them are linearly dependent. $V_R(t) = R\,i(t)$.

(iii) More than minimum no. of state variable can be defined.

<u>General state space Representation</u>

From above discussion an n^{th} order system having 'm' i/ps and 'p' o/ps can be represented as 'n' no of 1st order differential eqns (K/as state eqns) and 'p' no. of o/p equations.

Further, state eqns and o/p eqns will be linear combinations of all state variables and i/p variables.

State Equations:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1m}u_m$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2m}u_m$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nm}x_n$$

In vector form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

In Matrix form,

$$\boxed{\dot{X} = AX + BU} \quad \text{——(18)}$$

Eq^n-(18) is k/as ~~state~~ state eq^n.

X = state Vector
U = input vector
A = system matrix
B = input matrix

Similarly, o/p eq^n can be written as

$$y_1 = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots + d_{1m}u_m$$
$$y_2 = c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots + d_{2m}u_m$$
$$\vdots$$
$$y_p = c_{p1}x_1 + c_{p2}x_2 + \cdots + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \cdots + d_{pm}u_m$$

In vector form,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

In Matrix Form,

$$\boxed{Y = CX + DU} \quad \text{——(19)}$$

Eq^n-(19) is k/as Output equation.

Eq^n-(18) & (19) togeather is k/as state space representation.

Y = output vector
C = output Matrix
D = Feedforward Matrix.

State Variable:

It is the minimal set of linearly independent system variables such that knowledge these variables at $t = t_0$ along with forcing function for $t \geq t_0$, completely determines the behaviours of the system for $t \geq t_0$.

State Vector:

A vector whose components are state variable.

State space:

The n-dimensional space containing n-state variables as its axes.

STATE SPACE REPRESENTATION USING PHYSICAL VARIABLES

1. Electrical Network

For electrical network, current through inductor and voltage across capacitors are selected as states.

(a) Given the electrical network in figure-1, obtain the SS representation taking current through resistor and capacitor to be the output.

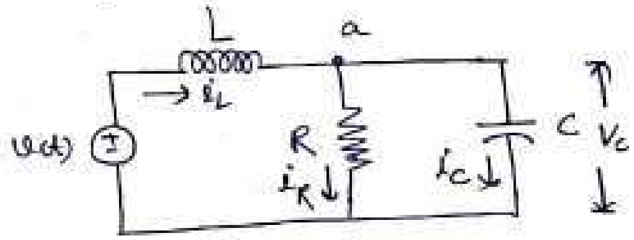


Figure-1

Step-1: Define state variables

$u(t) = v(t)$

$x_1 = i_L$

$x_2 = v_C$

$y_1 = i_R$

$y_2 = i_C$

Step-2: Form state eq^n

Applying KVL to mesh-1

$$v(t) = L\frac{di_L}{dt} + V_C \quad \text{--- (1)}$$

or, $$\frac{di_L}{dt} = -\frac{1}{L}V_C + \frac{1}{L}v(t) \quad \text{--- (2)}$$

Applying KCL at node a

$$i_L = i_R + i_C \quad \text{--- (3)}$$

$$i_L = \frac{V_C}{R} + C\frac{dv_C}{dt} \quad \text{--- (4)}$$

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or, $\frac{dV_c}{dt} = -\frac{1}{RC} V_c - \frac{1}{C} i_L$ —(5)

Step-3 : Form o/p eqⁿ

$i_R = \frac{V_c}{R}$ —— (6)

$i_C = C \frac{dV_c}{dt}$ — (7)

$= C \left(-\frac{1}{RC} V_c - \frac{1}{C} i_L \right) = -\frac{1}{R} V_c - i_L$

Step-4 :

Writing state eqⁿs & o/p eqⁿ in terms state variable & i/p variable we get.

$\dot{x}_1 = 0.x_1 - \frac{1}{L} x_2 + \frac{1}{L} u(t)$

$\dot{x}_2 = -\frac{1}{C} x_1 - \frac{1}{RC} x_2 + 0.u(t)$

or, $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ -\frac{1}{C} & \frac{1}{RC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u(t)$

$y_1 = 0.x_1 + \frac{1}{R} x_2 + 0.u(t)$

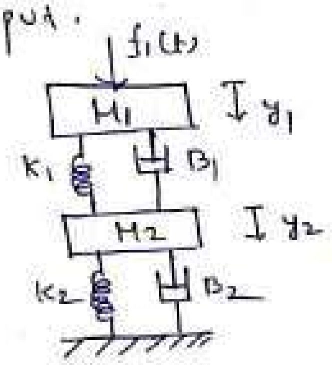
$y_2 = -x_1 - \frac{1}{R} x_2 + 0.u(t)$

or, $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/R \\ -1 & -1/R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$

(2) Mechanical system

For mechanical system position and velocity are selected as state variable of the system.

(b) Given the mechanical system in figure-2, obtain the SS representation taking y_2 to be the output.



Step 1: Define state variables

$x_1 = y_1$

$x_2 = y_2$

$x_3 = \frac{dy_1}{dt} = \dot{x}_1$

$x_4 = \frac{dy_2}{dt} = \dot{x}_2$

Step-2: Form state equation.

Writing force balance equation for mass M_1 & M_2

$$f(t) = M_1 \frac{d^2 y_1}{dt^2} + B_1 \frac{d}{dt}(y_1 - y_2) + K_1 (y_1 - y_2) \quad \text{---(1)}$$

$$0 = M_2 \frac{d^2 y_2}{dt^2} + B_1 \frac{d}{dt}(y_2 - y_1) + K_1 (y_2 - y_1) + B_2 \frac{dy_2}{dt} + K_2 y_2 \quad \text{---(2)}$$

Substituting state variables defined in step-1 in eqn-(1) & (2)

$$f(t) = M_1 \dot{x}_3 + B_1 (x_3 - x_4) + K_1 (x_1 - x_2) \quad \text{---(3)}$$

& $$0 = M_2 \dot{x}_4 + B_1 (x_4 - x_3) + K_1 (x_2 - x_1) + B_2 x_4 + K_2 x_2 \quad \text{---(4)}$$

$$\dot{x}_3 = -\frac{K_1}{M_1} x_1 + \frac{K_1}{M_1} x_2 - \frac{B_1}{M_1} x_3 + \frac{B_1}{M_1} x_4 + \frac{1}{M_1} f(t)$$

& $$\dot{x}_4 = \frac{K_1}{M_2} x_1 - \left(\frac{K_1 + K_2}{M_2}\right) x_2 + \frac{B_1}{M_2} x_3 - \left(\frac{B_1 + B_2}{M_2}\right) x_4$$

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

Step-3: (Form o/p eqn

$$y_1 = x_2$$

Step-4:

Writing state eqn & o/p in terms input variable & state variable we get,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_1}{M_1} & \frac{K_1}{M_1} & -\frac{B_1}{M_1} & \frac{B_1}{M_1} \\ \frac{K_1}{M_2} & -\left(\frac{K_1+K_2}{M_2}\right) & \frac{B_1}{M_2} & -\left(\frac{B_1+B_2}{M_2}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \end{bmatrix} f(t)$$

$$y_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Hence for a mechanical system
No. of state variable = 2 × order of the system.

(3) SS modelling of armature controlled DC Motor

The schematic of armature controlled DC Motor is depicted in figure-3.

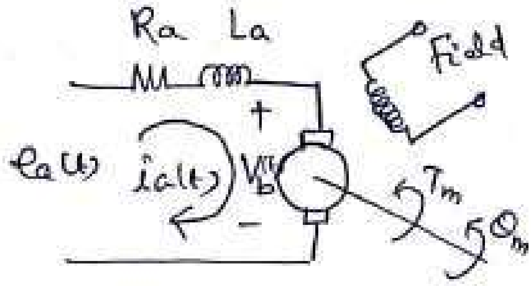


figure-3

Emf balance eqⁿ is

$$E_a = i_a R_a + L_a \frac{di_a}{dt} + k_b \frac{d\theta_m}{dt} \quad -(1)$$

Torque balance eqⁿ is

$$J \frac{d^2\theta_m}{dt^2} + B \frac{d\theta_m}{dt} = k_T i_a \quad -(2)$$

Choosing the states of the system to be

$$x_1 = \theta_m$$
$$x_2 = \frac{d\theta_m}{dt} = \dot{x}_1 \quad -(3)$$
$$x_3 = i_a$$

Substituting state variables in eqⁿ-(1) & (2) we get,

$$E_a = x_3 R_a + L_a \dot{x}_3 + K_b x_2 \quad (\text{from eq}^n\text{-}(1))$$

or, $$\dot{x}_3 = 0.x_1 - \frac{K_b}{L_a} x_2 - \frac{R_a}{L_a} x_3 + \frac{1}{L_a} E_a \quad -(4)$$

from eqⁿ-(2)

$$J \dot{x}_2 + B x_2 = k_T x_3$$

or, $$\dot{x}_2 = 0.x_1 - \frac{B}{J} x_2 + \frac{K_T}{J} x_3 + 0.E_a \quad -(5)$$

from eqⁿ (3), (4) & (5)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -B/J & K_T/J \\ 0 & -K_b/L_a & -R_a/L_a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} E_a$$

Selecting θ_m to be the output,

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

STATE SPACE REPRESENTATION USING PHASE VARIABLES

→ State space representation using physical variables are convinients especially when the actual system structure is known.

→ But if system system transfer function known (not the actual structure of the system) then state space representation can be obtained using phase (dual-phase) variable form.

1. PHASE VARIABLE FORM

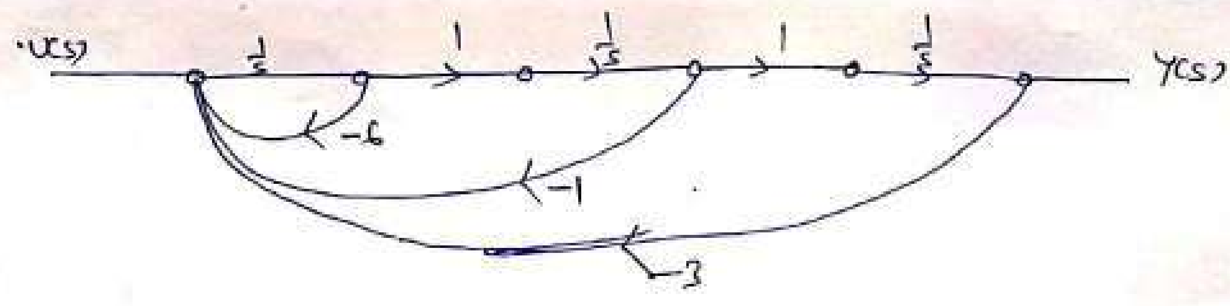
A rational TF can be synthesized using a simulation diagram that consists of integration blocks, constant blocks and summation operation.

Let's obtain the phase variable formulation of the following TF using simulation diagram.

$$T(s) = \frac{-5s^2+4s-12}{s^3+6s^2+s+3}$$

or, $$T(s) = \frac{-5/s + 4/s^2 + -12/s^3}{1-\left(\frac{6}{s}+\frac{1}{s^2}+3/s^3\right)} = \frac{P_1+P_2+P_3}{1-(L_1+L_2+L_3)}$$

The numerator term is represented by combining the o/ps of the integrators as shown above.

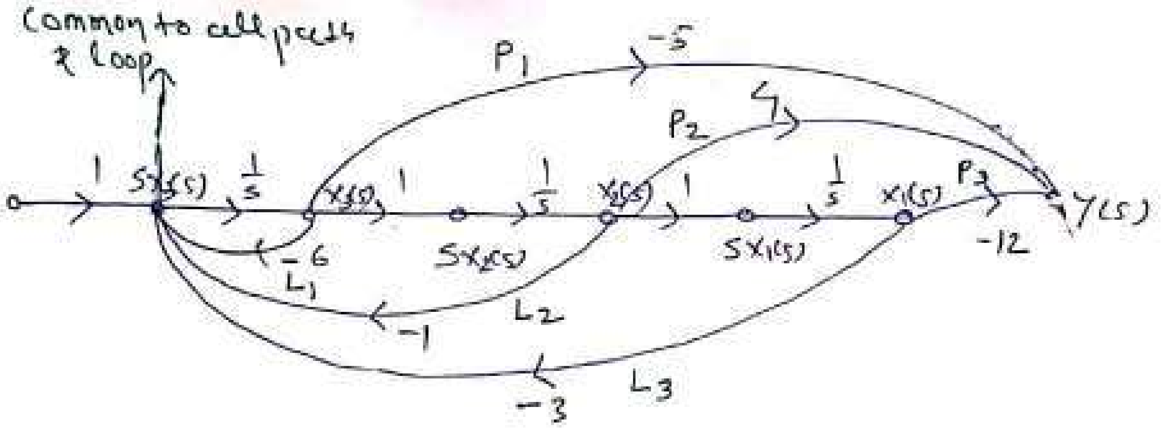


To avoid product of loop gain terms, all the three loops share a common starting point.

The output of each integrator represents a state variable (phase variable). Hence, phase variables are obtained from

One of the system variable (in this example the o/p variable) and its derivatives.

Now combining above two diagrams,



Hence, from the simulation diagram,

$$sX_1(s) = X_2(s)$$
$$sX_2(s) = X_3(s)$$
$$sX_3(s) = -3X_1(s) - X_2(s) - 6X_3(s) + U(s)$$
& $$Y(s) = -12X(s) + 4X_2(s) - 5X_3(s)$$

Converting above eqn into time domain,

$$\frac{dx_1}{dt} = x_2(t)$$
$$\frac{dx_2}{dt} = x_3(t)$$
$$\frac{dx_3}{dt} = -3x_1(t) - x_2(t) - 6x_3(t) + u(t)$$
& $$y(t) = -12x_1(t) + 4x_2(t) - 5x_3(t)$$

Expressing above eqn in matrix format

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -12 & 4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0.u$$

or more compactly, $\dot{x} = Ax + Bu$ & $y = Cx$

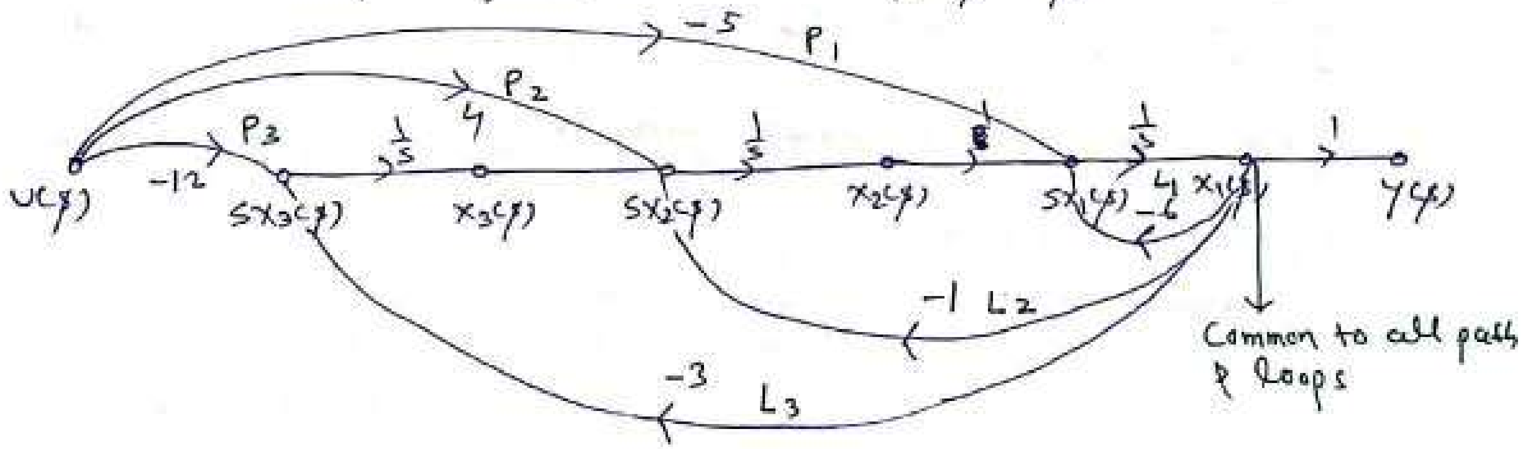
In the phase variable respreyentation (1) o/ps of all integrators are combined to form o/p. & (2) all loops connected to a node that connect input $U(s)$.

However, it is also possible to couple input to each of the integrator input and all loops touch a node that couples to the o/p node $Y(s)$. This form of presentation is called as "DUAL PHASE VARIABLE FORM".

(2) DUAL PHASE VARIABLE FORM

Solving above example,

$$T(s) = \frac{-5s^2+4s-12}{s^3+6s^2+s+3} = \frac{-5/s+4/s^2-12/s^3}{1-\left(\frac{6}{s}+\frac{1}{s^2}+\frac{3}{s^3}\right)} = \frac{P_1+P_2+P_3}{1-(L_1+L_2+L_3)}$$



From the simulation diagram,

$$sX_1(s) = -6X_1(s) + X_2(s) - 5U(s)$$
$$sX_2(s) = -X_1(s) + X_3(s) + 4U(s)$$
$$sX_3(s) = -3X_1(s) - 12U(s)$$
$$\& \quad Y(s) = X_1(s)$$

Converting above eqⁿ into time domain form,

$$\dot{x}_1(t) = -6x_1(t) + x_2(t) - 5u(t)$$
$$\dot{x}_2(t) = -x_1(t) + x_3(t) + 4u(t)$$
$$\dot{x}_3(t) = -3x_1(t) - 12u(t)$$
$$\& \quad y(t) = x_1(t)$$

Expressing in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 0 \\ -1 & 0 & 1 \\ -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -5 \\ 4 \\ -12 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + 0.u(t)$$

KEY OBSERVATION FROM ABOVE TWO FORM OF SS REPRESENTATION

PHASE VARIABLE FORM

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -1 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -12 & 4 & -5 \end{bmatrix} x + 0.u$$

DUAL PHASE VARIABLE FORM

$$\dot{x} = \begin{bmatrix} -6 & 1 & 0 \\ -1 & 0 & 1 \\ -3 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -5 \\ 4 \\ -12 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + 0.u.$$

Above two forms are derived from TF $\dfrac{Y(s)}{U(s)} = \dfrac{-5s^2 + 4s - 12}{s^3 + 6s^2 + s + 3}$

In phase variable form,

MATRIX A : Last row elements are negative of co-efficient of the TF denominator in ascending power of 's'.
The first column is all zero except last element
The remaining sub-matrix is an identity matrix

MATRIX B : Column vector of zeros except the last element which is 1.

MATRIX C : Row vector that contains numerator co-efficients in ascending power of 's'.

Dual phase variable form can be obtained from the phase variable form by substituting,
B with C, row with column, first with last & ascending with descending.

Because of the above-mentioned patterns, it is possible to obtain the SS representation in phase (dual-phase) variable form directly from the inspection of TF.

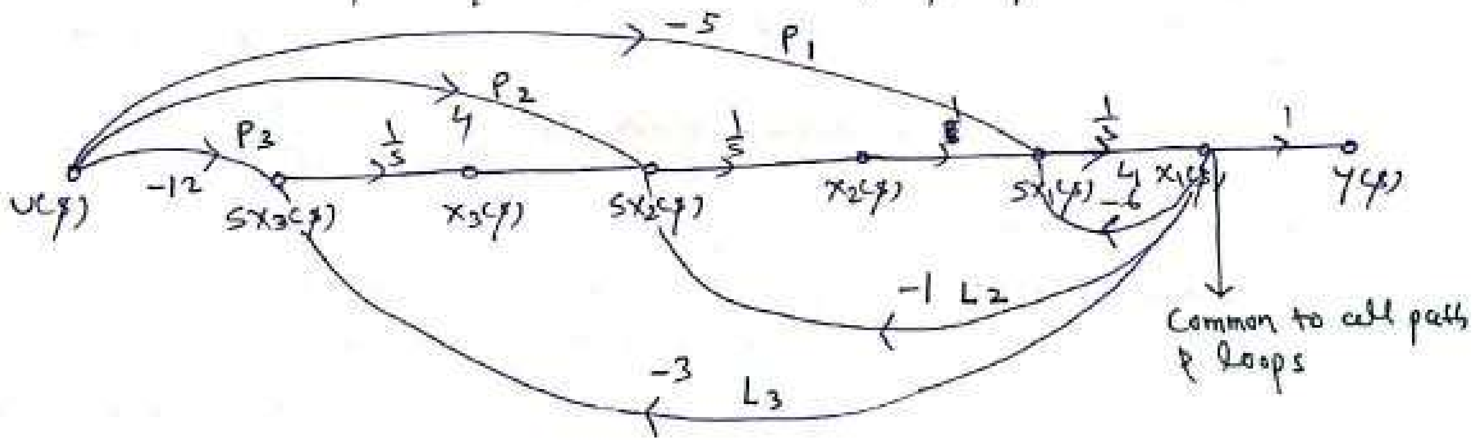
In the phase variable respreyentation (1) o/ps of all integrators are combined to form o/p. & (2) all loops connected to a node that connect input $U(s)$.

However, it is also possible to couple input to each of the integrator input and all loops touch a node that couples to the o/p node $Y(s)$. This form of presentation is called as "DUAL PHASE VARIABLE FORM".

(2) DUAL PHASE VARIABLE FORM

Solving above example,

$$T(s) = \frac{-5s^2+4s-12}{s^3+6s^2+s+3} = \frac{-5/s+4/s^2-12/s^3}{1-\left(\frac{6}{s}+\frac{1}{s^2}+\frac{3}{s^3}\right)} = \frac{P_1+P_2+P_3}{1-(L_1+L_2+L_3)}$$



From the simulation diagram,

$$sX_1(s) = -6X_1(s) + X_2(s) - 5U(s)$$
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Converting above eq^n into time domain form,

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$$\dot{x}_2(t) = -x_1(t) + x_3(t) + 4u(t)$$
$$\dot{x}_3(t) = -3x_1(t) - 12u(t)$$
$$\& \quad y(t) = x_1(t)$$

If a system TF is given by,

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

Then, phase variable form matrices are given by,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & 1 \\ -a_n & -a_{n-1} & \cdot & \cdot & \cdots & -a_1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [\, b_n \ b_{n-1} \ \cdots \ b_1 \,] \quad D = [0]$$

Dual-phase variable form matrices are given by,

$$A = \begin{bmatrix} -a_1 & 1 & 0 & & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & 0 & & \\ \vdots & & \vdots & & 0 \\ & & & & 1 \\ -a_n & 0 & 0 & & 0 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$C = [\, 1 \ 0 \ 0 \ \cdots \ 0 \,], \quad D = [0]$$

Matrices that have the special structure as observed in matrix A of phase & dual-phase variable form of representation are called as companion form.

An important ~~format~~ Property of companion matrix is, its charecteristic equation can be obtained by inspection.

As in this case, charecteristic eq^n of A matrices is

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n = 0$$

Advantages of PHASE VARIABLE form:

1. It provides a link between TF (Freq. domain) and statespace (Time domain)
2. SS model is intutive from the TF model.

Disadvantages of PHASE VARIABLE form:

1. Phase variable in general are not physical variable of the system & therefore are not available for direct measurement & control.

NOTE:

1. Phase variable forms are also called as BUSH form, CANONICAL form or COMPANION form.
2. Phase variable form $\longrightarrow$ Controlable Canonical form
   Dual phase variable form $\longrightarrow$ Observable canonical form

STATE TRANSFORMATIONS & DIAGONALISATION

As already been discussed that the state space representation of a system is not unique, by chosing different states for the system different state space realizations / representations can be obtained. These different realizations can be obtained by linear transformations of the states.

$x(t) = P\,z(t)$, where P = non-singular matrix
$x(t)$ = Old State
$z(t)$ = New state

$$\dot{x}(t) = Ax(t) + Bu(t) \Rightarrow P\dot{z} = APz + Bu$$

$$\text{or, } \dot{z} = P^{-1}APz + P^{-1}Bu = \tilde{A}z + \tilde{B}u$$

The o/p eq^n becomes,

$$y = Cx + Du = CPz + Du = \tilde{C}z + \tilde{D}u$$

Hence, $\tilde{A} = P^{-1}AP$, $\tilde{B} = P^{-1}B$, $\tilde{C} = CP$, $\tilde{D} = D$

Thus, using linear transformation $x = Pz$, different realizations can be obtained. The supermacy of different realizations may be accessed based on the numerical superiority they provide.

One of the numerical superiority can be to choose P matrix such that $\tilde{A} = P^{-1}AP$ becomes a diagonal matrix. A diagonal system matrix has the advantages of de-coupled state eq^n which lead to trivial solutions.

Choosing P matrix to be eigen vectors of the system will lead to $\tilde{A} = P^{-1}AP$ to be diagonal.

If a system TF is given by,

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

Then, phase variable form matrices are given by,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_1 \end{bmatrix} \quad D = [0]$$

Dual-phase variable form matrices are given by,

$$A = \begin{bmatrix} -a_1 & 1 & 0 & & 0 \\ -a_2 & 0 & 1 & & 0 \\ & & 0 & & \\ \vdots & & \vdots & & 0 \\ & & & & 1 \\ -a_n & 0 & 0 & & 0 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad D = [0]$$

Matrices that have the special structure as observed in matrix A of phase & dual-phase variable form of representation are called as companion form.

An important ~~form~~ Property of companion matrix is, its charecteristic equation can be obtained by inspection.

As in this case, charecteristic eq^n of A matrices is

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n = 0$$

Advantages of PHASE VARIABLE form:

1. It provides a link between TF (Freq. domain) and state space (Time domain)
2. SS model is intutive from the TF model.

Disadvantages of PHASE VARIABLE form:

1. Phase variable in general are not physical variable of the system & therefore are not available for direct measurement & control.

Diagonalization using Partial Fraction Expansion

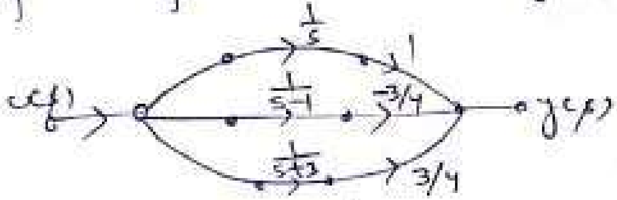
(a) Simple charecteristic Roots

$$T(s) = \frac{s^2 - s - 3}{s(s-1)(s+3)}$$

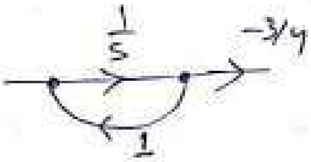
Expanding the TF using partial fraction expansion,

$$T(s) = \frac{1}{s} + \frac{-3/4}{s-1} + \frac{3/4}{s+3}$$

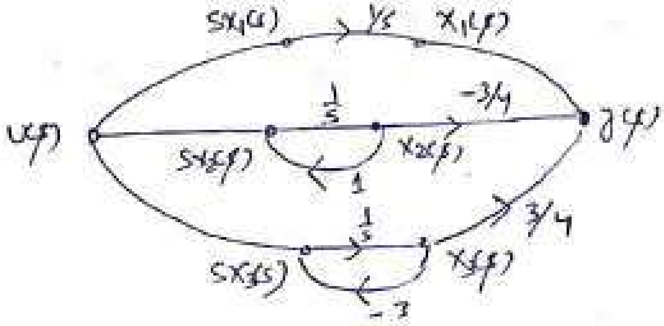
The above TF can be represented as parallel connection of 3 - first order subsystems as



$$-\frac{3}{4} \times \frac{1}{s-1} = \frac{\frac{1}{s}}{1 + \frac{1}{s} \times (-1)} \left(-\frac{3}{4}\right) =$$



$$\frac{3}{4} \times \frac{1}{s+3} = \frac{\frac{1}{s}}{1 + \frac{1}{s} \times 3} \times \frac{3}{4} =$$



$$sX_1(s) = U(s)$$
$$sX_2(s) = U(s) + X_2(s)$$
$$sX_3(s) = U(s) + (-3)X_3(s)$$
$$Y(s) = X_1(s) - \frac{3}{4}X_2(s) + \frac{3}{4}X_3(s)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -\frac{3}{4} & \cdot\frac{3}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

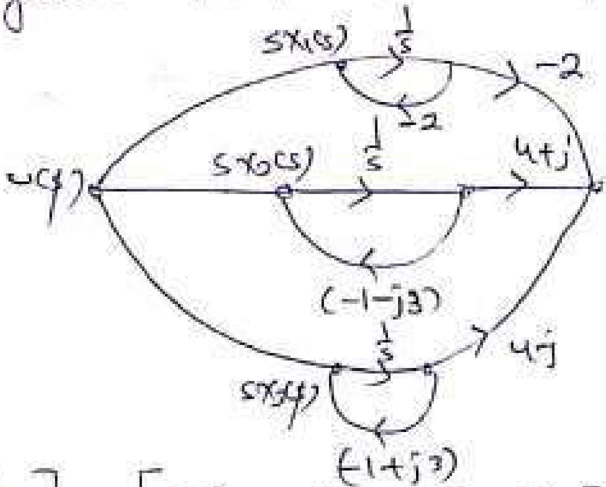
(b) Complex conjugate charecteristic Roots

$$T(s) = \frac{6s^2 + 26s + 8}{(s+2)(s^2+2s+10)}$$

Expanding above TF using partial fraction expansion

$$T(s) = \frac{-2}{s+2} + \frac{4+j}{s+1+j3} + \frac{4-j}{s+1-j3}$$

Above TF can be represented in diagonal form whose diagonal term will contain complex roots as shown below



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1-j3 & 0 \\ 0 & 0 & -1+j3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

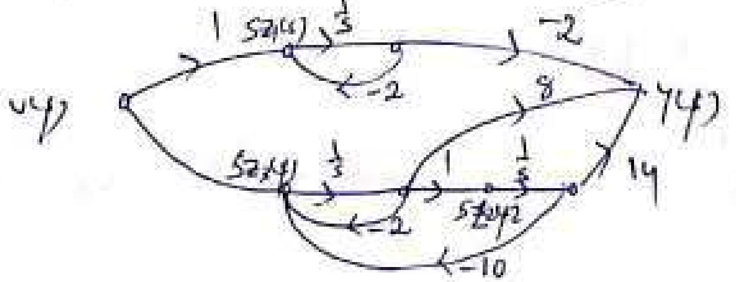
$$y = \begin{bmatrix} -2 & 4+j & 4-j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In order to obtain real roots, the complex sub-systems can be ~~obtain~~ combined and represented in phase variable form as shown below

i.e. $T(s) = \underbrace{\frac{-2}{s+2}}_{\text{Diagonal form}} + \underbrace{\frac{8s+14}{s^2+2s+10}}_{\text{phase variable form}}$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

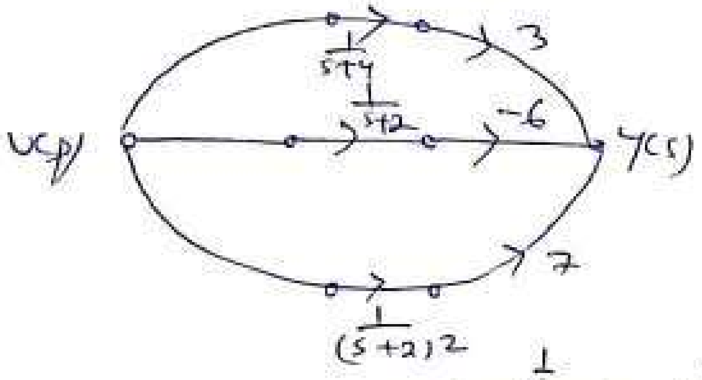
$$y = \begin{bmatrix} -2 & 14 & 8 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

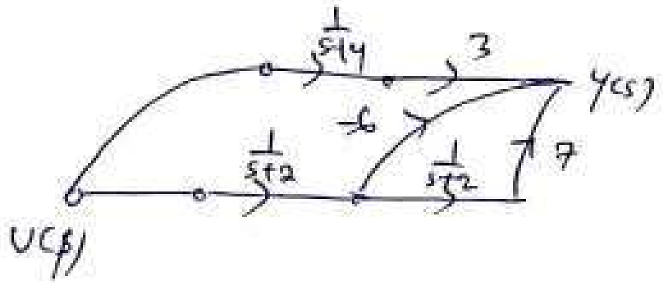


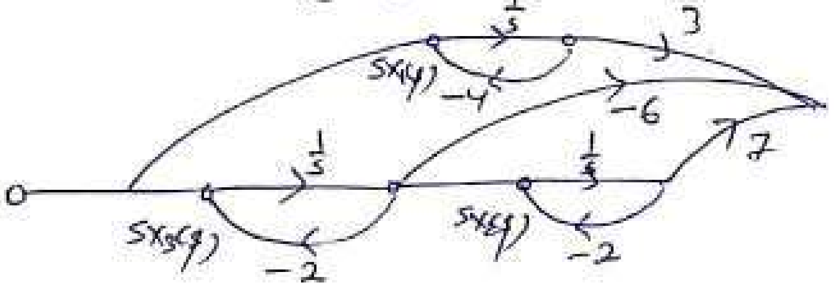
(3) Repeated characteristic Roots

The state equations for a system with repeated roots may be represented in "Jordan Cannonical form".

$$T(s) = \frac{10s^2 + 51s + 56}{(s+4)(s+2)^2} = \frac{3}{s+4} + \frac{-6}{s+2} + \frac{7}{(s+2)^2}$$







$$sX_1(s) = U(s) - 4X_1(s)$$
$$sX_2(s) = -2X_2(s) + X_3(s)$$
$$sX_3(s) = -2X_3(s) + U(s)$$
$$Y(s) = 3X_1(s) - 6X_3(s) + 7X_2(s)$$

or,
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Jordan cell

$$y = \begin{bmatrix} 3 & 7 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Concept of Eigen Values & Eigen Vectors

Eigen values of a state matrix corresponding to the state eqn of a state model are similar to the poles of a TF corresponding to root-locus model.

Hence eigen values are the set of roots, which may be obtained from the solution of characteristic eqn.

If λ represents the eigen values of a characteristic eqn, then the modified form of characteristic eqn may be

$$|\lambda I - A| = 0$$

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$$

Eigen vector (m_i) for a particular eigen value (λ_i) must satisfy the eqn

$$[\lambda_i I - A][m_i] = 0$$

Eigen vectors corresponding to a particular eigen value of the system represents, the directn in which the poles (eigen value) will be drifted when a sudden disturbance is applied to the system.

Combination of all the eigen vector in matrix form is called as modal matrix.

$$M = [m_1 : m_2 : \cdots m_n]$$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

Calculation of eigen vector

$$|\lambda I - A| = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{bmatrix}$$

$\lambda_1 = -1$, $\lambda_2 = -2$ and $\lambda_3 = -3$

m_1 corresponding to $\lambda_1 = -1$

$$|\lambda I - A| = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 6 & 11 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

m_2 corresponding to $\lambda_2 = -2$

$$\begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & -1 \\ 6 & 11 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 \\ -6 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

m_3 corresponding to $\lambda_3 = -3$

$$\begin{bmatrix} -3 & -1 & 0 \\ 0 & -3 & -1 \\ 6 & 11 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} +2 \\ -6 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$$

modal matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$

Q) Diagonalize the following system. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$ $C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

<u>Solⁿ:</u>

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Step-I

$$|\lambda I - A| = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{bmatrix} = \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$= (\lambda+1)(\lambda+2)(\lambda+3)$$

$$\lambda_1 = -1, \quad \lambda_2 = -2, \quad \lambda_3 = -3$$

Step-II

$$P = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$P^{-1}B = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ +3 \end{bmatrix}$$

$$CP = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$\dot{Z} = P^{-1}APZ + P^{-1}BU$$

$$y = CPZ$$

$$\dot{Z} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} u.$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Q) For the following system determine the transformation matrix for diagonalization.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & +3 \end{bmatrix}$$

Soln: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$

$$|\lambda I - A| = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

$$= \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 1$$

Step-II Since A has multiple root, so its transformation matrix will be, $S = \begin{bmatrix} 1 & 0 & 0 \\ \lambda_1 & 1 & 0 \\ \lambda_1^2 & 2\lambda_1 & 1 \end{bmatrix}$

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Step-III

INVARIANCE OF EIGENVALUES

To prove the invariance of the eigenvalues under a linear transformation we have to show that the characteristic polynomials $|\lambda I - A|$ and $|\lambda I - P^{-1}AP|$ are identical.

$$
\begin{aligned}
|\lambda I - P^{-1}AP| &= |\lambda P^{-1}P - P^{-1}AP| \\
&= |P^{-1}(\lambda I - A)P| \\
&= |P^{-1}||\lambda I - A||P| \\
&= |P^{-1}||P||\lambda I - A| = |P^{-1}P||\lambda I - A| \\
&= |\lambda I - A|
\end{aligned}
$$

SOLUTION OF LTI STATE EQUATION.

1. Solution of Homogenous state eqn:

A system can be represented as,

$$\dot{x} = Ax + Bu$$

If we take $u = 0 \Rightarrow \dot{x} = Ax$ —(1) (Homogenous eqn)

Let's assume a soln $x(t)$ of the form.

$$x(t) = b_0 + b_1 t + b_2 t^2 + \dots + b_k t^k \quad \text{—(2)}$$

By substituting the assumed soln in eqn -(1)

$$b_1 + 2b_2 t + 3b_3 t^2 + \dots + k b_k t^{k-1}$$
$$= A(b_0 + b_1 t + b_2 t^2 + \dots + b_k t^k)$$

$$b_1 = Ab_0$$

$$b_2 = \frac{1}{2}Ab_1 = \frac{1}{2}A^2 b_0$$

$$b_3 = \frac{1}{3}Ab_2 = \frac{1}{3\times 2}A^3 b_0 = \frac{1}{3!}A^3 b_0$$

$$\vdots$$

$$b_k = \frac{1}{k!}A^k b_0$$

The value of b_0 can be determined by substituting $t=0$ in the eqn - (2)

$$x(0) = b_0$$

Hence $x(t)$ can be written as,

$$x(t) = \left(I + At + \frac{1}{2!}A^2t^2 + \dots + \frac{1}{k!}A^kt^k \right) x(0)$$

$$= e^{At} x(0)$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

e^{At} is denoted as $\phi(t)$

$\phi(t)$ is k/as state transistion matrix. Because the information of the system at any time t can be derived by the transition or transformation of the initial condition.

$$\boxed{x(t) = \phi(t)\, x(0)}$$

PROPERTISE OF STATE TRANSITION MATRIX

(i) $\phi(0) = e^{A0} = I$

(ii) $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$

(iii) $\phi^{-1}(t) = \phi(-t)$

(iv) $\phi(t_1+t_2) = e^{A(t_1+t_2)} = e^{At_1}.e^{At_2} = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$

(v) $\phi(t)^n = \phi(nt)$

(vi) $\phi(t_2-t_1)\phi(t_1-t_0) = \phi(t_1-t_0)\phi(t_2-t_1) = \phi(t_2-t_0)$

LAPLACE TRANSFORM APPROACH TO THE SOLUTION OF HOMOGENOUS EQUATION

A system on SS can be defined as

$$\dot{x} = Ax$$

Taking Laplace Transform on both side,

$$sX(s) - X(0) = AX(s)$$

$$sX(s) - AX(s) = X(0)$$

$$(sI - A)X(s) = X(0)$$

Multiplying both sides by $(sI-A)^{-1}$,

$$X(s) = (sI-A)^{-1}X(0)$$

Taking Inverse Laplace transform,

$$\boxed{X(t) = \mathcal{L}^{-1}[(sI-A)^{-1}]\,X(0)}$$

We can write, $\phi(t) = \mathcal{L}^{-1}[(sI-A)^{-1}]$

Q) Find the state-transition matrix, $\phi(t)$ of the following system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Also determine $\phi^{-1}(t)$.

Soln: For the given system, $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

we known that $\phi(t) = e^{At} = \mathcal{L}^{-1}[(sI-A)^{-1}]$

$$|sI - A| = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$|sI - A|^{-1} = \frac{1}{s(s+3)+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

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$$(sI-A)^{-1} = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\mathcal{L}^{-1}\left[(sI-A)^{-1}\right] = \mathcal{L}^{-1}\begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

$$\phi^{-1}t = \phi(-t) = \begin{bmatrix} 2e^{t}-e^{2t} & e^{t}-e^{2t} \\ -2e^{t}+2e^{2t} & -e^{t}+2e^{2t} \end{bmatrix}$$

CAYLEY HAMILTON'S THEOREM

It states that, " the system matrix A satisfies its own charecteristic eq^n".

If charecteristic eq^n of the system is given by

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n = 0$$

then,
$$\boxed{A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1}A + a_n I = 0}$$

Proof:-

The charecteristic eq^n of the system is given by

$$|\lambda I - A| = 0$$

$$\Rightarrow \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$$

We can write,

$$\text{adj}\,|\lambda I - A| = B_1\lambda^{n-1} + B_2\lambda^{n-2} + \cdots + B_{n-1}\lambda + B_n$$

Again,
$$(\lambda I - A)^{-1} = \frac{\text{adj}(\lambda I - A)}{|\lambda I - A|}$$

$I\,|\lambda I - A| = \text{adj}(\lambda I - A)(\lambda I - A)$ [Multiplying $(\lambda I - A)$

$I\,|\lambda I - A| = (B_1\lambda^{n-1} + B_2\lambda^{n-2} + \cdots + B_{n-1}\lambda + B_n)(\lambda I - A)$

The above eq^n reduces to zero iff $|\lambda I - A| = 0$

$\Rightarrow \lambda = A$

Hence we obtain,

$$\boxed{A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I = 0}$$

Application of Cayley-Hamilton's Theorem

Find the polynomial, $N(A) = A^4 + A^3 + A^2 + A + I$ where $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Soln: $N(A) = A^4 + A^3 + A^2 + A + I$

In scalar form the previous eqn can be written as

$$N(s) = s^4 + s^3 + s^2 + s + 1 \quad -(1)$$

The charecteristic eqn of A

$$|sI - A| = 0 \Rightarrow \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = 0 \Rightarrow s^2 + 3s + 2 = 0$$

$$P(s) = s^2 + 3s + 2 \quad -(2)$$

Dividing N(s) by P(s) we obtain

$$s^2+3s+2 \,\big)\, s^4 + s^3 + s^2 + s + 1 \,\big(\, s^2 - 2s + 5$$

$$\cancel{s^4} + 3s^3 + 2s^2$$

$$(-)\quad(-)\quad(-)$$

$$-2\cancel{s^3} - s^2 + s + 1$$

$$-2\cancel{s^3} - 6s^2 - 4s$$

$$(+)\quad(+)\quad(+)$$

$$5\cancel{s^2} + 5s + 1$$

$$5s^2 + 15s + 10$$

$$(-)\quad(-)\quad(-)$$

$$-10s - 9$$

$$N(s) = Q(s)P(s) + R(s)$$

$$N(s) = (s^2 + 3s + 2)(s^2 - 2s + 5) + (-10s - 9)$$

In matrix form,

$$N(A) = (A^2 + 3A + 2I)(A^2 - 2A + 5I) + (-10A - 9I)$$

But as per cayley Hamilton's Thm,

$$A^2 + 3A + I = 0$$

$$\Rightarrow N(A) = R(A) = -10A - 9I$$

$$= -10\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -10 \\ 20 & 30 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -9 & -10 \\ 20 & 21 \end{bmatrix}$$

So any polynomial $N(s)$ can be written in the form

$N(s) = P(s)\cdot Q(s) + R(s) \Rightarrow N(s) = R(s)$ [As $P(s)=0$]

where $P(s)$ is the characteristic polynomial of A + $R(s)$ is a polynomial of the form

$$\boxed{R(s) = \alpha_0 + \alpha_1 s + \alpha_2 s^2 + \dots + \alpha_{n-1} s^{n-1}}$$

Q) Obtain the STM for the following system, using cayley Hamilton's theorem.

$$A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$$

Solⁿ: $|sI - A| = 0 \Rightarrow \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} = 0$

$$\begin{vmatrix} s & -2 \\ -1 & s+3 \end{vmatrix} = 0 \Rightarrow s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, s_2 = -2$$

Since the system matrix is of 2nd order, the polynomial $R(s)$ will be 1st order.

$R(s) = \alpha_0 + \alpha_1 s$ — (1)

but $N(s) = R(s)$, As per cayley Hamilton's Thm

$F(s_1) = R(s_1)$

$e^{s_1 t} = \alpha_0 + \alpha_1 s_1$

$e^{-t} = \alpha_0 - \alpha_1$ — (2)

$F(s_2) = R(s_2)$

$e^{s_2 t} = \alpha_0 + \alpha_2 s_2$

$e^{-2t} = \alpha_0 - 2\alpha_1$ — (3)

from eqⁿ (2) & (3)

$$\begin{aligned} \cancel{\alpha_0} - \alpha_1 &= e^{-t} \\ \cancel{\alpha_0} - 2\alpha_1 &= e^{-2t} \\ (-) \quad (+) \quad &\quad (-) \\ \hline \alpha_1 &= e^{-t} - e^{-2t} \end{aligned}$$

Substituting the value of α_1 in eqⁿ – (2)

$$\alpha_0 = e^{-t} + (e^{-t} - e^{-2t}) = 2e^{-t} - e^{-2t}$$

$R(s) = \alpha_0 + \alpha_1 s$

$= (2e^{-t} - e^{-2t}) + (e^{-t} - e^{-2t}) s$

$R(A) = \alpha_0 I + \alpha_1 A$

$= (2e^{-t} - e^{-2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t}) \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

$= \begin{bmatrix} 2e^{-t} - e^{-2t} & 0 \\ 0 & 2e^{-t} - e^{-2t} \end{bmatrix} + \begin{bmatrix} 0 & -2e^{-t} + 2e^{-2t} \\ e^{-t} - e^{-2t} & -3e^{-t} + 3e^{-2t} \end{bmatrix}$

$\phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & -2e^{-t} - e^{-2t} \\ e^{-t} - e^{-2t} & 2e^{-2t} - e^{-t} \end{bmatrix}$

$R(A) = F(A) = e^{At}$

Determination of STM by power series Method

Q) For the following system, obtain STM by power series method. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Soln: $\phi(t) = e^{At} = \sum_{K=0}^{\infty} \frac{A^K t^K}{K!}$

Approximating upto 4th term ie $K = 0, 1, 2 \,\&\, 3$

$\phi(t) = e^{At} = I + At + \frac{1}{2} A^2 t^2 + \frac{1}{6} A^3 t^3 + \cdots$

Here $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$\phi(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} t + \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} t^2 + \frac{1}{6} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} t^3 + \cdots$

$= \begin{bmatrix} \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots\right) & 0 \\ \left(t + t^2 + \frac{t^3}{2} + \cdots\right) & \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots\right) \end{bmatrix}$

$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$

Q(1) Find e^{At} using Caley. Hamilton's Theorem.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

step-1 compute the eigen values

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{bmatrix}$$

$$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_1 = -1 \ \& \ \lambda_2 = -1$$

step-2

Since order of the given matrix is 2. Reminder polynomial will be of 1st order.

Let's choose a 1st remainder polynomial

$$R(\lambda) = \alpha_0 + \alpha_1 \lambda .$$

step-3

Since we have to compute e^{At},

$$f(A) = e^{At}$$

then $f(\lambda) = e^{\lambda t}$

for $\lambda_1 = -1$, $f(\lambda) = e^{\lambda t} = f(-1) = e^{-t}$

$$R(\lambda) = \alpha_0 + \alpha_1 \lambda$$

$$R(-1) = \alpha_0 - \alpha_1$$

$$\Rightarrow \alpha_0 - \alpha_1 = e^{-t} \quad \text{—— (1)}$$

for $\lambda_1 = -1$

$$\frac{d}{d\lambda} f(\lambda) \Big|_{\lambda = \lambda_2} = t e^{\lambda t}$$

$$\Rightarrow \frac{d}{d\lambda} f(\lambda) \Big|_{\lambda = -1} = t e^{-t}$$

Similarly, $\frac{d}{d\lambda} R(\lambda) = \alpha_1$

$\Rightarrow \alpha_1 = te^{-t}$ — (2)

Substituting eq^n-(2) in eq^n-(1)

$$\alpha_0 = \alpha_1 + e^{-t} = te^{-t} + e^{-t} = (1+t)e^{-t}$$

Hence $R(\lambda) = \alpha_0 + \alpha_1 \lambda$

$$= (1+t)e^{-t} + te^{-t}\lambda$$

Therefore $R(A) = (1+t)e^{-t} I + te^{-t} A$

$$(1+t)e^{-t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{-t}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

Q(2) Compute A^{10} using Cayley-Hamilton's Theorem.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Sol^n: $\lambda_1 = -1$, $\lambda_2 = -2$

$R(\lambda) = \alpha_0 + \alpha_1 \lambda$

$f(\lambda_1) = (-1)^{10} = \alpha_0 - \alpha_1$

$f(\lambda_2) = (-2)^{10} = \alpha_0 - 2\alpha_1$

$\Rightarrow \alpha_0 = -1022$ & $\alpha_1 = -1023$

$R(A) = \alpha_0 I + \alpha_1 A$

$$= \begin{bmatrix} -1022 & -1023 \\ 2046 & 2047 \end{bmatrix}$$

MINIMAL POLYNOMIAL THM

As per Cayley-Hamilton's Thm, every system matrix of A of order $n \times n$ satisfies its own characteristic eqn. But however the characteristic eqn is not necessarily the polynomial of least order having A as root.

So the polynomial of least order having A as root is k/as minimal polynomial.

→ In an $n \times n$ matrix A has n distinct eigen values, then the minimal polynomial of A is identical to the characteristic ~~eqn~~ polynomial.

→ If the multiple eigen values of the system matrix A are linked in Jordan chain, the minimal polynomial and the characteristic polynomial are identical.

→ If however, the multiple eigen values of A are not linked in a Jordan chain, the minimal polynomial is of lower degree than the characteristic polynomial.

<u>Procedure for determining minimal polynomial</u>

Let the minimal polynomial $\phi(\lambda)$ be represented by

$$\phi(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m \quad , m \le n$$

$$\phi(A) = A^m + a_1 A^{m-1} + \cdots + a_{m-1} A + a_m I = 0$$

$$\phi(\lambda) = \frac{|\lambda I - A|}{d(\lambda)}$$

(i) Form adj$(\lambda I - A)$ and write the elements of adj$(\lambda I - A)$ as factored polynomials in λ

(ii) Determin $d(\lambda)$ as the greatest common divisor of all the elements of adj$(\lambda I - A)$. Choose the co-efficent of the highest degree terms in λ of $d(\lambda)$ to be 1.

If there is no common divisor, $d(\lambda) = 1$

(iii) The minimal polynomial $\phi(\lambda)$ is then given as

$$\boxed{\phi(\lambda) = \frac{|\lambda I - A|}{d(\lambda)}}$$

Q) If a system is represented by the system matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

determine the minimal polynomial and prove minimal polynomial theorem.

Soln:

Step-1 (calculate $\lambda I - A$)

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda-2 & 0 \\ 0 & -3 & \lambda-1 \end{bmatrix}$$

$$|\lambda I - A| = (\lambda-2)(\lambda-2)(\lambda-1)$$

$$\operatorname{adj}(\lambda I - A) = \begin{bmatrix} (\lambda-2)(\lambda-1) & 0 & 0 \\ 0 & (\lambda-2)(\lambda-1) & 0 \\ 0 & 3(\lambda-2) & (\lambda-2)^2 \end{bmatrix}$$

$$d(\lambda) = \lambda - 2$$

$$\text{Hence } \phi(\lambda) = \frac{|\lambda I - A|}{d(\lambda)} = \frac{(\lambda-2)^2(\lambda-1)}{(\lambda-2)}$$

$$\phi(\lambda) = (\lambda-2)(\lambda-1) = \lambda^2 - 3\lambda + 2$$

$$\phi(A) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 9 & 1 \end{bmatrix} - 3\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Note:

for $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $|\lambda I - A| = (\lambda-2)^2(\lambda-1)$

for $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $|\lambda I - A| = (\lambda-2)^2(\lambda-1)$

Calculation of e^{At}

If a system is represented as,

$\dot{x} = Ax$, then its soln is given by

$$\boxed{x(t) = e^{At}x(0)} \quad \text{—— (1)}$$

If we apply a transformation, to transform it into DCF

$x = Pz$

$\dot{z} = P^{-1}APz = Dz$

So, the soln of the system will be,

$z(t) = e^{Dt}z(0)$

But $z(t) = P^{-1}x(t)$ and $z(0) = P^{-1}x(0)$

$P^{-1}x(t) = e^{Dt}P^{-1}x(0)$

multiplying P on both side,

$$\boxed{x(t) = Pe^{Dt}P^{-1}x(0)} \quad \text{—— (2)}$$

Comparing eqn-(1) and (2), we got,

$$\boxed{e^{At} = Pe^{Dt}P^{-1}}$$

Similarly, if a system matrix can be transformed into Jordan-Canonical form by suitable transformation matrix S then we can calculate e^{At} as

$$\boxed{e^{At} = Se^{Jt}S^{-1}}$$

NOTE

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}, \quad e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}, \lambda_1 = \lambda_2, \lambda_3$$

$$e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{1}{2}t^2 e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$

Q) Calculate e^{At} for the system represented by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Solⁿ:

Step-1

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 0 & \lambda+2 \end{bmatrix}$$

$$|\lambda I - A| = \lambda^2 + 2\lambda$$

Hence charecteristic eqⁿ is $\lambda^2 + 2\lambda = 0$

Eigen values are, $\lambda_1 = 0$ and $\lambda_2 = -2$

Step-2

since the system has distinc eigen value so we have to calculate P and e^{Dt}.

$$P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-2} \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & +\frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^0 & 0 \\ 0 & e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

so, $e^{At} = P e^{Dt} P^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

SYLVESTER'S POLYNOMIAL

The sylvester's polynomial of system having $n \times n$ system matrix A is given by,

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{m-1} & e^{\lambda_1 t} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{m-1} & e^{\lambda_2 t} \\ \vdots & & & & & \\ 1 & \lambda_m & \lambda_m^2 & \cdots & \lambda_m^{m-1} & e^{\lambda_m t} \\ I & A & A^2 & & A^{m-1} & e^{At} \end{vmatrix} = 0$$

Q) Calculate the state-transition matrix for the following system using sylvester's polynomial / sylvester's interpolation method.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

<u>soln</u>: The sylvester's polynomial for the given system is

$$\begin{vmatrix} 1 & \lambda_1 & e^{\lambda_1 t} \\ 1 & \lambda_2 & e^{\lambda_2 t} \\ I & A & e^{At} \end{vmatrix} = 0$$

For the system, $\lambda_1 = 0$ and $\lambda_2 = -2$

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & -2 & e^{-2t} \\ I & A & e^{At} \end{vmatrix} = 0$$

$$-2e^{At} - Ae^{-2t} + A + 2I = 0$$

$$e^{At} = \frac{1}{2}\left[A + 2I - Ae^{-2t}\right]$$

$$= \frac{1}{2}\left[\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} e^{-2t}\right]$$

$$= \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

Solution of Non-Homogenous State Equations

Let us consider the non-homogeneous state eq^n

$$\dot{X} = AX + BU$$

$$\dot{X} - AX = BU$$

Multiplying both side of the eq^n by e^{-At},

$$e^{-At}[\dot{X} - AX] = e^{-At} BU$$

$$\Rightarrow \frac{d}{dt}[e^{-At} X] = e^{-At} BU$$

Integrating the previous eq^n between 0 & t

$$e^{-At} X = \int_0^t e^{-A\tau} BU d\tau + X(0)$$

$$X(t) = e^{At} X(0) + \int_0^t e^{A(t-\tau)} BU(\tau) d\tau$$

$$\boxed{X(t) = \underbrace{\phi(t) X(0)}_{\text{Zero i/p component}} + \underbrace{\int_0^t \phi(t-\tau) BU(\tau) d\tau}_{\text{Zero state component}}}$$

Laplace Transform Approach :-

$$\dot{X}(t) = AX(t) + BU(t)$$

Taking Laplace Transform on both side,

$$sX(s) - \cancel{X(0)} = AX(s) + BU(s)$$

$$(sI - A)X(s) = X(0) + BU(s)$$

Multiplying both sides by $(sI-A)^{-1}$

$$X(s) = (sI - A)^{-1} X(0) + (sI - A)^{-1} BU(s)$$

Taking inverse Laplace Transform,

$$\mathcal{L}^{-1} X(s) = \mathcal{L}^{-1}[(sI-A)^{-1} X(0)] + \mathcal{L}^{-1}[(sI-A)^{-1} BU(s)]$$

$$\boxed{x(t) = \phi(t) x(0) + \int_0^t e^{A(t-\tau)} BU(\tau) d\tau}$$

NOTE : If the initial cond^n is k/at $t = t_0$, then

$$\boxed{X(t) = \phi(t-t_0) X(t_0) + \int_{t_0}^t \phi(t-\tau) B u(\tau) d\tau}$$

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Q) Obtain the responce of the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where $u(t)$ is the unit step function occuring at $t=0$, or $u(t)=1(t)$

Sol^n: For the given system,

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The state transistion matrix $\phi(t) = e^{At}$

$$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$e^{A(t-\tau)} = \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix}$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B\, u(\tau)$$

$$= e^{At}x(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1]\, d\tau$$

$$= e^{At}x(0) + \int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} d\tau$$

$$= e^{At}x(0) + \begin{bmatrix} -e^{-(t-\tau)} + \frac{1}{2}e^{-2(t-\tau)} \\ e^{-(t-\tau)} - e^{-2(t-\tau)} \end{bmatrix}_0^t$$

$$= e^{At}x(0) + \begin{bmatrix} -e^{0} + \frac{1}{2}e^{0} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{0} - e^{0} - e^{-t} + e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix} \quad \text{As } x_1(0) = 0 \;\&\; x_2(0) = 0$$

CONTROLLABILITY

A system is said to be controllable at time t_0 if it is possible by means of an control vector to transfer the system from any initial state $x(t_0)$ to any other state in a finite interval of time.

A system is said to be complete state controllable if $M = [B \quad AB \quad A^2B \cdots\cdots \quad A^{n-1}B]$ has rank n.
This matrix is K/as controllability matrix.

In practical design of control system, it is essential that the o/p of the system should be controllable rather than the state of the system.

So the condn for o/p controllability can be

$$[CB \quad CAB \quad CA^2B \cdots \quad CA^{n-1}B \quad D]$$

has rank m. where $A \rightarrow n \times n$
$B \rightarrow n \times r$
$C \rightarrow m \times n$
$D \rightarrow m \times r$

OBSERVABILITY

A system is said to be observable at time t_0 if, with the system in state $x(t_0)$, it is possible to determine this state from the observation of the o/p over a finite time interval.

A system is said to be completely observable if

$$N = [C^* \quad A^*C^* \quad (A^*)^2C^* \cdots (A^*)^{n-1}C^*]$$

has rank n. This matrix is called as observability matrix.

Q) Investigate the controllability and observability of the following system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Soln:

- For the given system

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

→ Condn for complete state controllability

$[B \;\; AB]$ should have rank 2

$$AB = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Controllability matrix is given by

$$M = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad |M| = -1, \text{ Hence it has rank 2}$$

So, the system is completely state controllable.

→ Condn for complete o/p controllability

$[CB \;\; CAB]$ should have rank 1

$$CB = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$CAB = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$$[CB \;\; CAB] = \begin{bmatrix} 0 & 1 \end{bmatrix} \;:\; \text{rank } 1$$

So, the given system completely o/p controllable.

→ Condn for complete oberservability

$[C^* \;\; A^* C^*]$ should have rank 2.

$$C^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A^* C^* = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad |N| = 1, \quad \text{rank} = 2$$

So, the system is completely state observable.

NOTE :-

(i) Cond^n for complete state controllability in s-plane :-

The necessary and sufficient condition for complete state controllability in s-plane is, there should not any cancellation occure in the transfer function or transfer matrix. If cancellation occures, then the system conn't be controlled in the direction of the canceled mode.

(ii) Cond^n for complete state observability in s-plane :-

The necessary and sufficient cond^n for complete state observability in s-plane is, there should not any cancellation occure in the transfer func^n or transfer matrix. If any cancellation occures then the system is not observable in the direc^n of the canceled mode.

(iii) Stabilizability :-

For a partially controllable system, if the uncontrollable modes are stable and unstable modes are controllable, then the system is said to be stabilizable.

(iv) Detectability :-

For a partially observable system, if the unobservable modes are stable and unstable modes are observable, then the system is said to detectable.

Effect of Pole-zero Cancellation

Q) Check the controllability of the system in s-plane.

$$\frac{X(s)}{U(s)} = \frac{(s+2.5)}{(s+2.5)(s-1)}$$

Sol^n : $$\frac{X(s)}{U(s)} = \frac{(s+2.5)}{(s+2.5)(s-1)} = \frac{s+2.5}{s^2+1.5s-2.5}$$

Representing the system in OCF

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2.5 & -1.5 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2.5 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2.5 \\ 1 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2.5 \\ 1 \end{bmatrix} u$$

$$A = \begin{bmatrix} 0 & 2.5 \\ 1 & -1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 2.5 \\ 1 & -1.5 \end{bmatrix} \begin{bmatrix} 2.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$$

$$M = [B \quad AB] = \begin{bmatrix} 2.5 & 2.5 \\ 1 & 1 \end{bmatrix}$$

$|M| = 0$

Hence the rank of the matrix is 1. So the system is not completely state controllable. This is also concluded from its TF, that due to cancellation of the factor $(s+2.5)$ it can't completely state controllable.

Causes of Uncontrollability

(i) Pole-zero cancellation in TF model.

(ii) Symmetry in the system.

i.e. $\dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$

$y = [1 \quad 1] x$

(iii) Use of redundant state variables in the modelling.

(iv) In appropriate or insufficient control actuators or sensors.