Lecture Notes

On

Advance Control Systems

7Th Semester Electrical Engineering

Part 4: Modeling and Analysis of Digital Control Systems

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Course Objective:

The objective of this course is to equip students with a deep understanding of discrete-time control systems, state variable analysis, and nonlinear system behavior. Students will learn to analyze, design, and implement control strategies using modern techniques, including Z-transform methods, state-space representations, and Lyapunov stability analysis.

Course Outcome:

- **Analyze**: Analyze discrete-time and continuous-time control systems using Z-transform and state-space methods to determine system behavior.
- **Design**: Design control systems utilizing feedback strategies, pole placement, and observer design to achieve specified performance criteria.
- **Evaluate**: Evaluate the stability of linear and nonlinear systems using Routh's criterion and Lyapunov's methods, assessing their robustness.
- **Apply**: Apply techniques for modeling and simulating nonlinear systems, including phase plane and describing function methods, to solve practical engineering problems.

Z- TRANSFORM

Z - Transform

Let $f(k)$ denotes a real value sequence, $f(0), f(1), f(2) \cdots$ or equivalently $f(k)$, for $k = 0, 1, 2, 3 \cdots$

Then Z - transform of the function $f(k)$ can be defined as,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

where, z is a complex variable, defined by $z = u + jw$

One - sided Z-transform

Generally practical signals does not exists for $k < 0$.

so, $$\boxed{Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)\, z^{-k}}$$

is $k \not< 0$ s one-sided or single sided Z - transform.

Z - transform of some common functions

(i) Unit step

$$f(k) = \begin{cases} 1, & k \geq 0 \\ 0 & k < 0 \end{cases}$$

On discretizing unit step function,

$$f(kT) = \begin{cases} 1, & kT \geq 0 \\ 0, & kT < 0 \end{cases}$$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(kT)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} 1 \cdot z^{-k}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$\boxed{F(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}}$$

Z- transform is a mathematical tool used to analyze discrete time system.

Z - transform of a time function $x(t)$ is evaluated at sampling instants. i.e. $x(T), x(2T), x(3T) \cdots$ where T is the sampling period.

$$X(z) = z[x(t)] = z[x(kT)]$$
$$= \sum_{k=0}^{\infty} x(kT)\, z^{-k}$$

(ii) Ramp function

$$f(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$f(KT) = \begin{cases} KT, & KT \geq 0 \\ 0, & KT < 0 \end{cases}$$

$$F(z) = \sum_{K=0}^{\infty} f(KT) z^{-K} = \sum_{K=0}^{\infty} (KT) \cdot z^{-K}$$

$$= 0 + Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + \cdots$$

$$= Tz^{-1}(1 + 2z^{-1} + 3z^{-2} + \cdots)$$

But from binomial expansion Th^{m},

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \cdots \frac{n(n+1)\cdots(n+r-1)}{r!} + \cdots$$

so, $(1-z^{-1})^{-2} = 1 + 2z^{-1} + 3z^{-2} + \cdots$

$$F(z) = Tz^{-1} \times (1-z^{-1})^{-2} = \frac{Tz^{-1}}{(1-z^{-1})^2}$$

$$\boxed{F(z) = \frac{Tz}{(z-1)^2}}$$

(iii) Exponential Function

a) Exponential increasing function

$$f(t) = \begin{cases} e^{at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$f(KT) = \begin{cases} e^{aKT}, & KT \geq 0 \\ 0, & KT < 0 \end{cases}$$

$$F(z) = \sum_{K=0}^{\infty} f(KT) \cdot z^{-K} = \sum_{K=0}^{\infty} e^{aKT} \cdot z^{-K}$$

$$= \sum_{K=0}^{\infty} (e^{aT} z^{-1})^{K}$$

$$\boxed{\text{NOTE:-} \quad \sum_{K=0}^{\infty} c^{K} = \frac{1}{1-c}}$$

$$F(z) = \frac{1}{1 - e^{aT} z^{-1}}$$

$$= \frac{z}{z - e^{aT}}$$

For exponential decaying function

$$f(t) = \begin{cases} e^{-at} & , t \geq 0 \\ 0 & , t < 0 \end{cases} , \quad f(KT) = \begin{cases} e^{-aKT} & , KT \geq 0 \\ 0 & , KT < 0 \end{cases}$$

$$F(z) = \sum_{K=0}^{\infty} f(KT) \cdot z^{-K} = \sum_{K=0}^{\infty} e^{-aKT} z^{-K}$$

$$= \sum_{K=0}^{\infty} (e^{-aT} z^{-1})^K$$

$$= \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}}$$

(iv) Sinusoidal function

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

$$Z (e^{j\omega t}) = \frac{z}{z - e^{j\omega T}}$$

Substituting $e^{j\omega t} = \cos \omega t + j \sin \omega t$

$$Z (e^{j\omega t}) = \frac{z}{z - (\cos \omega T + j \sin \omega T)}$$

$$= \frac{z}{(z - \cos \omega T) - j \sin \omega T}$$

$$= \frac{z\{(z - \cos \omega T) + j \sin \omega T\}}{\{(z - \cos \omega T) - j \sin \omega T\}\{(z - \cos \omega T) + j \sin \omega T\}}$$

$$= \frac{z(z - \cos \omega T) + j z \sin \omega T}{(z - \cos \omega T)^2 + (\sin \omega T)^2}$$

But $Z(e^{j\omega t}) = Z(\cos \omega t + j \sin \omega t)$

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So, $$\mathcal{Z}(\cos\omega t) = \frac{z(z-\cos\omega T)}{(z-\cos\omega T)^2 + (\sin\omega T)^2}$$

$$\boxed{\mathcal{Z}(\cos\omega T) = \frac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}}$$

Similarly,

$$\boxed{\mathcal{Z}(\sin\omega T) = \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}}$$

Propertise of z-transform

(i) Linearity Property

$$\mathcal{Z}[a f(k) + b g(k)] = a F(z) + b G(z)$$

Proof: $$\mathcal{Z}[a f(k) + b g(k)] = \sum_{k=0}^{\infty} (a f(k) + b g(k)) z^{-k}$$

$$= \sum_{k=0}^{\infty} a f(k) \cdot z^{-k} + \sum_{k=0}^{\infty} b g(k) z^{-k}$$

$$= a \sum_{k=0}^{\infty} f(k) z^{-k} + b \sum_{k=0}^{\infty} g(k) z^{-k}$$

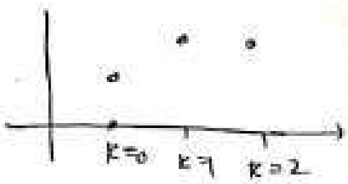
$$= a F(z) + b G(z)$$

(ii) Shifting Property

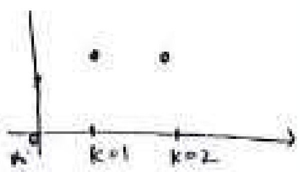
Left shift (advanced)

A sequence $f(k)$ when shifted one interval to the left can be written as

$$g(k) = f(k+1), \quad k \geq -1$$



$f(k)$



$f(k+1)$

$$\mathcal{Z}\{f(k+1)\} = \sum_{k=0}^{\infty} f(k+1) \cdot z^{-k}$$

$$= z \sum_{k=0}^{\infty} f(k+1) z^{-(k+1)}$$

Let $k+1 = m$

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$$Z\{f(k+1)\} = z\sum_{m=1}^{\infty} f(m)\,z^{-m}$$

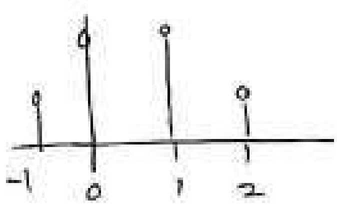
$$= z\left[\sum_{m=0}^{\infty} f(m)z^{-m} - f(0)\right]$$

$$= zF(z) - zf(0)$$

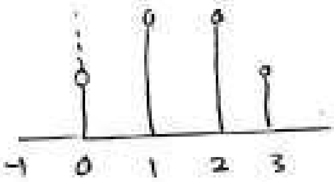
In general,

$$\boxed{Z[f(k+n)] = z^n F(z) - \sum_{l=0}^{n-1} f(l)\,z^{n-l}, \quad k \geq -n}$$

<u>Right Shift (delayed)</u>



$f(k)$



$f(k-1)$

A sequence $f(k)$ when shifted to right one interval,

$$g(k) = f(k-1);\quad k \geq 1$$

$$Z[f(k-1)] = \sum_{k=0}^{\infty} f(k-1)\cdot z^{-k}$$

$$= z^{-1}\sum_{k=0}^{\infty} f(k-1)\, z^{-(k-1)}$$

Let, $k-1 = m$

$$Z[f(k-1)] = z^{-1}\sum_{m=-1}^{\infty} f(m)\cdot z^{-m}$$

$f(m) = 0$ for $m < 0$

$$= z^{-1}\sum_{m=0}^{\infty} f(m)\,z^{-m}$$

$$= z^{-1}F(z)$$

In general,

$$\boxed{Z[f(k-n)] = z^{-n}F(z), \quad k \geq n}$$

(iii) Multiplication by K,

$$Z[k^n f(k)] = \left(-z\frac{d}{dz}\right)^n F(z)$$

Proof:

Let $Z f(k) = F(z)$

So, $Z[k f(k)] = \sum_{k=0}^{\infty} k f(k) \cdot z^{-k}$

$$= -z \sum_{k=0}^{\infty} -k f(k) \cdot z^{-k-1}$$

$$= -z \sum_{k=0}^{\infty} f(k) \cdot \frac{d}{dz} z^{-k} = -z \frac{d}{dz} \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= -z \frac{d}{dz} F(z)$$

In general, $Z\{k^n f(k)\} = \left(-z\frac{d}{dz}\right)^n F(z)$

(iv) Scaling property

$$Z[a^k f(k)] = F(z/a)$$

Proof:

$$Z[a^k f(k)] = \sum_{k=0}^{\infty} a^k f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k) \left(\frac{z}{a}\right)^{-k} = F\left(\frac{z}{a}\right)$$

(v) Initial Value Th^m

$$f(0) = \lim_{z \to \infty} F(z)$$

Proof:

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k} = f(0) + f(1) z^{-1} + f(2) z^{-2} + \cdots$$

Taking the limit, as $z \to \infty$, we obtain,

$$f(0) = \lim_{z \to \infty} F(z)$$

(v) Final Value ~~Convolution~~ Th^m

$$F(\infty) = \lim_{z\to 1} (1-z^{-1}) F(z)$$
$$= \lim_{z\to 1} (z-1) F(z)$$

If $F(z)$ is analytic for $|z|>1$

Proof:

$$Z[x(k)] = X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$
$$Z[x(k-1)] = z^{-1}X(z) = \sum_{k=0}^{\infty} x(k-1) z^{-k}$$
$$\sum_{k=0}^{\infty} x(k) z^{-k} - \sum_{k=0}^{\infty} x(k-1) z^{-k} = X(z) - z^{-1}X(z) = (1-z^{-1})X(z)$$
$$\lim_{z\to 1}\left[\sum_{k=0}^{\infty} x(k) z^{-k} - \sum_{k=0}^{\infty} x(k-1) z^{-k}\right] = \lim_{z\to 1} (1-z^{-1})X(z)$$

$\Rightarrow [x(0)-x(-1)]+[x(1)-x(0)]+[x(2)-x(1)] + \cdots = x(\infty) = \lim_{k\to\infty} x(k)$

(vi) Convolution Theorem

$$Z[h(k-m) * r(m)] = H(z)R(z)$$

Proof:

Let $C_k = h(k) * r(k) = \sum_{m=0}^{\infty} h(k-m) r(m)$

$$Z[c(k)] = C(z) = \sum_{k=0}^{\infty} c(k) z^{-k}$$
$$= \sum_{k=0}^{\infty}\left[\sum_{m=0}^{\infty} h(k-m) r(m)\right] z^{-k}$$

Interchanging the order of summation,

$$C(z) = \sum_{m=0}^{\infty} r(m) \sum_{k=0}^{\infty} h(k-m) \cdot z^{-k}$$

Let $j = k-m$,

$$C(z) = \sum_{m=0}^{\infty} r(m) \sum_{j=-m}^{\infty} h(j) z^{-j-m}$$
$$= \sum_{m=0}^{\infty} r(m) z^{-m} \sum_{j=0}^{\infty} h(j) z^{-j} \quad (\text{since } h(j)=0,\ j<0$$
$$= R(z)H(z)$$

Inverse Z-transform

It is a process of determining the sequence which generates the given z-transform.

If $F(z)$ be the z-transform of the sequence of $f(k)$

$$f(k) = Z^{-1}[F(z)]$$

Q(1) Obtain the z-transform of $\frac{1}{s^2(s+1)} = X(s)$

Ans: $X(s) = \frac{1}{s^2(s+1)}$

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$$

$$x(t) = (t - 1 + e^{-t})\,u(t)$$

$$x(kT) = (kT - 1 + e^{-kT})\,u(kT)$$

$$Z(x(kT)) = Z\left[(kT - 1 + e^{-kT})\,u(kT)\right]$$

$$= \frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}$$

$$= \frac{z\,(e^{-Tz} + 1 - 2e^{-T})}{(z-1)^2\,(z-e^{-T})}$$

Q(2) Determine the inverse z-transform of the following functions.

(i) $F(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$

(ii) $F(z) = \frac{z^2}{z^2 - z + 0.5}$

(iii) $F(z) = \frac{3z^2 + 2z + 1}{z^2 - 3z + 2}$

(i) $F(z) = \dfrac{1}{1-1.5z^{-1}+0.5z^{-2}}$

$$= \frac{1}{1-\frac{1.5}{z}+\frac{0.5}{z^2}} = \frac{z^2}{z^2-1.5z+0.5}$$

$$= \frac{z^2}{(z-1)(z-0.5)}$$

$$\frac{F(z)}{z} = \frac{z}{(z-1)(z-0.5)}$$

$$\frac{F(z)}{z} = \frac{A}{z-1} + \frac{B}{z-0.5}$$

$A = \dfrac{1}{1-0.5} = \dfrac{1}{0.5} = 2$

$B = \dfrac{(0.5)}{0.5-1} = -1$

$$\frac{F(z)}{z} = \frac{2}{z-1} + \frac{-1}{z-0.5}$$

$$F(z) = \frac{2z}{z-1} + \frac{-z}{z-0.5}$$

Taking inverse z-transform

$f(k) = 2u(k) - (0.5)^k$

$Z(a^k) = \dfrac{z}{z-a}$

$Z(1) = \dfrac{z}{z-1} u(k)$

(ii) $F(z) = \dfrac{z^2}{z^2-z+0.5}$

$$\frac{F(z)}{z} = \frac{z}{z^2-z+0.5} = \frac{z}{(z-0.5-j0.5)(z-0.5+j0.5)}$$

$$\frac{F(z)}{z} = \frac{A}{(z-0.5-j0.5)} + \frac{A^*}{(z-0.5+j0.5)}$$

$$\left[A = \frac{z}{\cancel{(z-0.5-j0.5)}} \times \frac{\cancel{(z-0.5-j0.5)}}{z-0.5+j0.5} \Bigg|_{z=0.5+j0.5} \right]$$

$$= 0.5 - j0.5$$

$$A^* = 0.5 + j0.5$$

$$\frac{F(z)}{z} = \frac{0.5-j0.5}{z-0.5-j0.5} + \frac{0.5+j0.5}{z-0.5+j0.5}$$

$$F(z) = 0.5-j0.5 \times \left(\frac{z}{z-0.5-j0.5}\right) + 0.5+j0.5\left(\frac{z}{z-0.5+j0.5}\right)$$

Taking inverse z-transform

$$f(k) = (0.5 - j0.5)(0.5 + j0.5)^k + (0.5 + j0.5)(0.5 - j0.5)^k$$

iii) $F(z) = \dfrac{3z^2 + 2z + 1}{z^2 - 3z + 2}$

Ans:

$$z^2 - 3z + 2 \;\Big)\; 3z^2 + 2z + 1 \;\Big(\; 3$$
$$\underline{\overset{(-)}{3z^2} \overset{(+)}{-9z} \overset{-}{+6}}$$
$$11z - 5$$

$$F(z) = 3 + \frac{11z - 5}{z^2 - 3z + 2}$$

$$= 3 + \frac{11z - 5}{(z-1)(z-2)}$$

$$= 3 + \frac{A}{(z-1)} + \frac{B}{z-2}$$

$$A = \frac{11 \times 1 - 5}{1 - 2} = \frac{6}{-1} = -6$$

$$B = \frac{11 \times 2 - 5}{2 - 1} = 17$$

$$= 3 + \frac{-6}{z-1} + \frac{17}{z-2}$$

$$F(z) = 3 - \frac{6}{z-1} + \frac{17}{z-2}$$

$$F(z) = 3 - \frac{1}{z}\frac{6z}{z-1} + \frac{1}{z}\frac{17z}{z-2}$$

$$= 3 - 6\, z^{-1}\left(\frac{z}{z-1}\right) + 17\, z^{-1}\left(\frac{z}{z-2}\right)$$

$$f(k) = 3 - 6\; 1.u(k-1) + 17\; 2^{k-1}$$

Q(3) Find the one-sided z-transform of the following sequences.

(i) t^2

(ii) $\sin \omega t$

(iii) $e^{at} \cos \omega t$

(i) $f(t) = t^2$

$f(kT) = (kT)^2 = k^2 T^2 = k^2 g(k)$

$g(z) = Z\{g(k)\} = z(T^2) = \sum_{k=0}^{\infty} T^2 z^{-k}$

$= T^2 \sum_{k=0}^{\infty} (z^{-1})^k = T^2 \times \frac{z}{z-1}$

$g(z) = \frac{T^2 z}{z-1}$

But $F(z) = k^2 g(z)$

$k^2 g(z) = \left(-z \frac{d}{dz}\right)^2 g(z)$

$= -z \frac{d}{dz}\left[-z \frac{d}{dz} \frac{T^2 z}{z-1}\right]$

$= -z \frac{d}{dz}\left[-z \times \left[\frac{T^2(z-1) - T^2 z}{(z-1)^2}\right]\right]$

$= -z \frac{d}{dz}\left[\frac{z T^2}{(z-1)^2}\right]$

$= -z \left[\frac{T^2(z-1)^2 - zT^2 \times 2(z-1)}{(z-1)^4}\right]$

$= -z \times \cancel{(z-1)}\left[\frac{T^2(z-1) - 2T^2 z}{(z-1)^{\cancel{4}\,3}}\right]$

$= -z \left[\frac{T^2 z - T^2 - 2T^2 z}{(z-1)^3}\right]$

$= \frac{(T^2 z + T^2) z}{(z-1)^3} = \frac{zT^2(1+z)}{(z-1)^3}$

(ii) $f(t) = \sin \omega t$

$f(kT) = \sin k\omega T$

$Z\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$

$F(z) = \sum_{k=0}^{\infty} \sin k\omega T \cdot z^{-k}$

But $\sin\omega kT = \dfrac{e^{j\omega kT} - e^{-j\omega kT}}{2j}$

So, $F(z) = \sum_{k=0}^{\infty} \dfrac{e^{j\omega kT} - e^{-j\omega kT}}{2j} z^{-k}$

$$= \frac{1}{2j}\left[\sum_{k=0}^{\infty} e^{j\omega kT} z^{-k} - \sum_{k=0}^{\infty} e^{-jk\omega T} z^{-k}\right]$$

$$= \frac{1}{2j}\left[\frac{z}{z - e^{j\omega T}} - \frac{z}{z - e^{-j\omega T}}\right]$$

$$= \frac{1}{2j}\left[\frac{z(z - e^{-j\omega T}) - z(z - e^{j\omega T})}{(z - e^{j\omega T})(z - e^{-j\omega T})}\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{z^2} - ze^{-j\omega T} - \cancel{z^2} + ze^{j\omega T}}{z^2 - ze^{-j\omega T} - ze^{j\omega T} + e^{j\omega T}\cdot e^{-j\omega T}}\right]$$

$$= \frac{z\left(\dfrac{e^{j\omega T} - e^{-j\omega T}}{2j}\right)}{z^2 - z(e^{j\omega T} + e^{-j\omega T}) + e^0}$$

$$= \frac{z\left(\dfrac{e^{j\omega T} - e^{-j\omega T}}{2j}\right)}{z^2 - 2z\left(\dfrac{e^{j\omega T} + e^{-j\omega T}}{2}\right) + 1}$$

$$= \frac{z \sin\omega T}{z^2 - 2z\cos\omega T + 1}$$

(iii) $f(t) = e^{-at}\cos\omega t$

$f(kT) = e^{-akT}\cos\omega kT$

$F(z) = \sum_{k=0}^{\infty} e^{-akT}\left(\dfrac{e^{j\omega kT} + e^{-j\omega kT}}{2}\right) z^{-k}$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} e^{-akT}\cdot e^{j\omega kT}\cdot z^{-k} + \sum_{k=0}^{\infty} e^{-akT} e^{-j\omega kT}\cdot z^{-k}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty}\left(e^{-aT}\cdot e^{j\omega T} z^{-1}\right)^{k} + \sum_{k=0}^{\infty}\left(e^{-aT} e^{-j\omega T} z^{-1}\right)^{k}\right]$$

$$= \frac{1}{2}\times\frac{1}{1-e^{-aT}e^{j\omega T}z^{-1}} + \frac{1}{2}\times\frac{1}{1-e^{-j\omega T}e^{-aT}z^{-1}}$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{e^{j\omega T}}{z e^{aT}}} + \frac{1}{1-\frac{e^{-j\omega T}}{e^{aT} z}}\right]$$

$$= \frac{1}{2}\left[\frac{z e^{aT}}{z e^{aT}-e^{j\omega T}} + \frac{z e^{aT}}{z e^{aT}-e^{-j\omega T}}\right]$$

$$= \frac{1}{2}\left[\frac{z e^{aT}\left(z e^{aT}-e^{-j\omega T}\right) + z e^{aT}\left(z e^{aT}-e^{j\omega T}\right)}{\left(z e^{aT}-e^{j\omega T}\right)\left(z e^{aT}-e^{-j\omega T}\right)}\right]$$

$$= \cancel{\frac{1}{2}\left[z^{2}e^{2aT} - z e^{-ja\omega T^{2}}\right.}$$

$$= \frac{1}{2}\, z e^{aT}\left[\frac{z e^{aT}-e^{-j\omega T} + z e^{aT}-e^{j\omega T}}{\left(z e^{aT}-e^{j\omega T}\right)\left(z e^{aT}-e^{-j\omega T}\right)}\right]$$

$$= \frac{1}{2}\, z e^{aT}\left[\frac{2z e^{aT}-\left(e^{j\omega T}+e^{-j\omega T}\right)}{z^{2}e^{2aT}-z e^{aT}\cdot e^{-j\omega T}-z e^{aT}\cdot e^{j\omega T}+e^{j\omega T-j\omega T}}\right]$$

$$= \frac{1}{2}\, z e^{aT}\left[\frac{2z e^{aT}-2\left(\frac{e^{j\omega T}+e^{-j\omega T}}{2}\right)}{z^{2}e^{2aT}-2z e^{aT}\left(\frac{e^{j\omega T}+e^{-j\omega T}}{2}\right)+1}\right]$$

$$= z e^{aT}\left[\frac{z e^{aT}-\cos\omega T}{z^{2}e^{2aT}-2z\cos\omega T+1}\right]$$

$$= \frac{z\left(z-e^{-aT}\cos\omega T\right)}{z^{2}e^{aT}-2z e^{-aT}\cos\omega T+e^{-aT}}$$

soln: $c(k+2) + 3c(k+1) + 4c(k) = r(k+1) - r(k)$

Taking z-transform on both sides of the above eqn and neglecting the initial condns we get

$$z^{2}C(z) + 3z^{1}C(z) + 4C(z) = z^{1}R(z) - R(z)$$

$$C(z)\left[z^{2} + 3z^{1} + 4\right] = R(z)(z-1)$$

$$\boxed{\frac{C(z)}{R(z)} = \frac{z-1}{z^{2}+3z+4}}$$

$$H(z) = \frac{z-1}{z^{2}+3z+4}$$

The waiting sequence is the impulse response $h(k)$ of the system.

So $h(k) = \mathcal{Z}^{-1}[H(z)]$

$$H(z) = \frac{z-1}{z^{2}+3z+4} = \frac{z-1}{\left(z+\frac{3}{2}+j\frac{\sqrt{7}}{2}\right)\left(z+\frac{3}{2}-j\frac{\sqrt{7}}{2}\right)}$$

$$= \frac{A}{\left(z+\frac{3}{2}+j\frac{\sqrt{7}}{2}\right)} + \frac{A^{*}}{\left(z+\frac{3}{2}-j\frac{\sqrt{7}}{2}\right)}$$

Applying partial fraction,

$$A = \frac{1}{2} - j\frac{5}{2\sqrt{7}}, \quad A^{*} = \frac{1}{2} + j\frac{5}{2\sqrt{7}}$$

$$H(z) = \frac{\left(\frac{1}{2} - j\frac{5}{2\sqrt{7}}\right)}{\left(z+\frac{3}{2}+j\frac{\sqrt{7}}{2}\right)} + \frac{\left(\frac{1}{2} - j\frac{5}{2\sqrt{7}}\right)}{\left(z+\frac{3}{2}-j\frac{\sqrt{7}}{2}\right)}$$

$$h(k) = z^{-1}[H(z)]$$

$$= \left(\frac{1}{2} - j\frac{5}{2\sqrt{7}}\right)\left(-\frac{3}{2} - j\frac{\sqrt{7}}{2}\right)^{k-1} u(k-1) + \left(\frac{1}{2} + j\frac{5}{2\sqrt{7}}\right) \times \left(-\frac{3}{2} + j\frac{\sqrt{7}}{2}\right)^{k-1} u(k-1)$$

COMPUTATION OF INVERSE Z-TRANSFORM USING INVERSION INTEGRAL METHOD

Inversion integral for the z-transform $X(z)$ is given by

$$Z^{-1}[X(z)] = x(kT) = x(k) = \frac{1}{2\pi j}\oint_c X(z)\, z^{k-1}\, dz$$

Inverse z-transform in terms of residues can be derived by using theory of complex variables.

$$x(kT) = x(k) = K_1 + K_2 + \cdots + K_m$$
$$= \sum_{i=1}^{m} \text{residue of } X(z)z^{k-1}\Big|_{z=z_i}.$$

Where, $K_1, K_2 \ldots K_m$ denotes residue of $X(z)z^{k-1}$ at poles $z_1, z_2 \ldots z_m$ respectively.

Case-1 If denominator of $X(z)z^{k-1}$ contains simple pole at $z = z_i$ then,

$$K = \lim_{z \to z_i} \left[(z - z_i)\, X(z)\, z^{k-1}\right]$$

Case-2 If denominator of $X(z)z^{k-1}$ contains multiple pole of order q then

$$K = \frac{1}{(q-1)!} \lim_{z \to z_i} \frac{d^{q-1}}{dz^{q-1}} \left[(z - z_i)^q\, X(z)\, z^{k-1}\right]$$

NOTE: Inversion integral method is a simple technique of evaluating inverse z-transform if, $X(z)z^{k-1}$ does not contain any pole at origin. $(z = 0)$.

Example 1:

Obtain $x(kT)$ using the inversion integral method when

$$X(z) = \frac{z(1 - e^{-aT})}{(z-1)(z - e^{-aT})}$$

Soln:
$$X(z)z^{k-1} = \frac{(1 - e^{-aT})\, z^k}{(z-1)(z - e^{-aT})}$$

For $k = 0, 1, 2 \cdots$ $X(z)z^{k}$ has two simple poles $z = z_1 = 1$ and $z = z_2 = e^{-aT}$.

$$x(k) = \sum_{i=1}^{2} \text{residue of } \frac{(1-e^{-aT})z^k}{(z-1)(z-e^{-aT})} \Bigg|_{z=z_i}$$

$$= K_1 + K_2$$

where, $K_1 = \lim_{z \to 1} \left[\cancel{(z-1)} \frac{(1-e^{-aT})z^k}{\cancel{(z-1)}(z-e^{-aT})} \right] = 1$

$$K_2 = \lim_{z \to e^{-aT}} \left[\cancel{(z-e^{-aT})} \frac{(1-e^{-aT})z^k}{(z-1)\cancel{(z-e^{-aT})}} \right]$$

$$= \frac{(1-e^{-aT})e^{-akT}}{(e^{-aT}-1)} = -e^{-akT}$$

Hence, $x(kT) = K_1 + K_2 = 1 - e^{-akT}$, $k = 0, 1, 2 \ldots$

<u>Example-2:</u>

Obtain inverse z-transform of $X(z) = \dfrac{z^2}{(z-1)^2(z-e^{-aT})}$

Soln: $X(z)z^{k-1} = \dfrac{z^{k+1}}{(z-1)^2(z-e^{-aT})}$

For $k = 0, 1, 2$, $X(z)z^{k-1}$ has a simple pole at $z = z_1 = e^{-aT}$ and a double pole at $z = z_2 = 1$

$$x(k) = \sum_{i=1}^{2} \left[\text{residue of } \frac{z^{k+1}}{(z-1)^2(z-e^{-aT})} \right]_{z=z_i}$$

$$= K_1 + K_2$$

$$K_1 = \lim_{z \to e^{-aT}} \left[\cancel{(z-e^{-aT})} \frac{z^{k+1}}{(z-1)^2\cancel{(z-e^{-aT})}} \right]$$

$$= \frac{e^{-aT(k+1)}}{(e^{-aT}-1)^2} = \frac{e^{-aT(k+1)}}{(1-e^{-aT})^2} = \frac{e^{-aT}e^{-akT}}{(1-e^{-aT})^2}$$

$$K_2 = \lim_{z \to 1} \frac{d}{dz} \left[\cancel{(z-1)^2} \frac{z^{k+1}}{\cancel{(z-1)^2}(z-e^{-aT})} \right]$$

$$= \lim_{z \to 1} \frac{d}{dz} \left[\frac{(k+1)z^k(z-e^{-aT}) - z^{k+1}}{(z-e^{-aT})^2} \right]$$

$$= \frac{(k+1)(1-e^{-aT}) - 1}{(1-e^{-aT})^2} = \frac{k - ke^{-aT} + \cancel{1} - e^{-aT} \cancel{-1}}{(1-e^{-aT})^2}$$

$$= \frac{K(1-e^{-aT}) - e^{-aT}}{(1-e^{-aT})^2} = \frac{K}{(1-e^{-aT})} - \frac{e^{-aT}}{(1-e^{-aT})^2}$$

Here, $x(K) = K_1 + K_2$

$$= \frac{e^{-aT} \cdot e^{-akT}}{(1-e^{-aT})^2} + \frac{K}{(1-e^{-aT})} - \frac{e^{-aT}}{(1-e^{-aT})^2}$$

$$= \frac{K}{(1-e^{-aT})} - \frac{e^{aT}(1-e^{-akT})}{(1-e^{-aT})^2}, \quad K = 0, 1, 2 \ldots$$

<u>Example 3:</u>

Using the inversion of integral method obtain the z-transform of $X(z) = \frac{10}{(z-1)(z-2)}$

Sol^n: $X(z)z^{k-1} = \frac{10z^K}{z(z-1)(z-2)}$

For, $K=0$, $x(z)z^{k-1}$ becomes

$$x(z)z^{k-1} = \frac{10}{z(z-1)(z-2)}$$

Here, for $K=0$, $x(z)z^{k-1}$ has three simple poles i.e. at $z = z_1 = 0$, $z = z_2 = 1$ and $z = z_3 = 2$.

Hence, for $K=0$,

$$x(0) = \sum_{i=1}^{3} \text{residue of } \left.\frac{10}{z(z-1)(z-2)}\right]_{\text{at } z = z_i}$$

$$= K_1 + K_2 + K_3$$

where $K_1 = \lim_{z \to 0} \left[\cancel{z} \times \frac{10}{\cancel{z}(z-1)(z-2)} \right] = 5$

$$K_2 = \lim_{z \to 1} \left[\cancel{(z-1)} \frac{10}{z\cancel{(z-1)}(z-2)} \right] = -10$$

$$K_3 = \lim_{z \to 2} \left[\cancel{(z-2)} \frac{10}{z(z-1)\cancel{(z-2)}} \right] = 5$$

Hence, $x(0) = K_1 + K_2 + K_3 = 5 - 10 + 5 = 0$

For, $K = 1, 2, 3,$

$$x(k) = \sum_{i=1}^{2} \left[\text{residue of } \frac{10 z^{k-1}}{(z-1)(z-2)} \right]_{z=z_i}$$

$$= K_1 + K_2$$

where $K_1 = \lim_{z \to 1} \left[\cancel{(z-1)} \times \frac{10 z^{k-1}}{\cancel{(z-1)}(z-2)} \right] = \frac{10}{-1} = -10$

$$K_2 = \lim_{z \to 2} \left[\cancel{(z-2)} \frac{10 z^{k-1}}{(z-1)\cancel{(z-2)}} \right] = \frac{10 \times 2^{k-1}}{2-1} = 10 2^{k-1}$$

Thus, $x(k) = K_1 + K_2 = -10 + 10 2^{k-1} = 10(2^{k-1} - 1)$ for $k = 1, 2, 3 \ldots$

$$x(k) = \begin{cases} 0 & \text{for } k = 0 \\ 10(2^{k+1} - 1) & \text{for } k = 1, 2, 3 \ldots \end{cases}$$

State Variable Analysis of Linear Discrete Time Systems

The general form of state model for multivariable LDS is

$$x(k+1) = A\,x(k) + B\,u(k)$$
$$y(k) = C\,x(k) + D\,u(k)$$

Q) Derive the ss model of a LDS represented by

$$y(k+2) - 1.7\,y(k+1) + 0.72\,y(k) = u(k).$$

Soln: Since it is a 2nd order difference eqn

$$x_1(k) = y(k)$$
$$x_2(k) = x_1(k+1) = y(k+1)$$

$$x_2(k+1) - 1.7\,x_2(k) + 0.72\,x_1(k) = u(k)$$
$$\Rightarrow x_2(k+1) = u(k) - 0.72\,x_1(k) + 1.7\,x_2(k)$$

so we can write,

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = -0.72\,x_1(k) + 1.7\,x_2(k) + u(k).$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Q) Derive the ss model of a LDS represented as

$$H(z) = \frac{z^3 + 8z^2 + 17z + 8}{z^3 + 6z^2 + 11z + 6}$$

Soln:

$$\frac{z^3 + 8z^2 + 17z + 8}{z^3 + 6z^2 + 11z + 6} = 1 + \frac{2z^2 + 6z + 2}{z^3 + 6z^2 + 11z + 6}$$

$$= 1 - \frac{1}{z+1} + \frac{2}{z+2} + \frac{1}{z+3}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + u(k)$$

Transfer function of a Linear Discrete System (LDS)

The TF of LDS is given by its Z-transform of its impulse response.

The TF of LDS system is also called as Z-transfer function or pulse transfer function.

If $h(k)$: Impulse response of LDS / waiting sequence.

$H(z) \leftrightarrow Z(h(z))$

$$\boxed{H(z) = \frac{C(z)}{R(z)}}$$

where $C(z) \longleftrightarrow Z(c(k))$

$R(z) \longleftrightarrow Z(r(k))$

Linear Difference Equation

Discrete time systems with sampled inputs giving sampled outputs are described by linear constant co-efficient difference equations.

$$a_0 c(k) + a_1 c(k-1) + a_2 c(k-2) + \cdots + a_{n-1} c(k-n+1) + a_n c(k-n)$$
$$= b_0 r(k) + b_1 r(k-1) + \cdots + b_{m-1} r(k-m+1) + b_m r(k-m)$$

where $c(k)$ is the output sequence & $r(k)$ is the i/p sequence.

Taking z-transform on both sides of the eq^n & neglecting initial conditions we get

$$a_0 C(z) + a_1 z^{-1} C(z) + a_2 z^{-2} C(z) + \cdots + a_{n-1} z^{-(n-1)} C(z) + a_n z^{-n} C(z)$$
$$= b_0 R(z) + b_1 z^{-1} R(z) + \cdots + b_{m-1} R(z) z^{-(m-1)} + b_m R(z) z^{-m}$$

The TF is given by,

$$\boxed{H(z) = \frac{C(z)}{R(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{m-1} z^{-(m-1)} + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{n-1} z^{-(n-1)} + a_n z^{-n}} \quad m \le n}$$

Q) The i/p – o/p relationship of a sample data system is described by $c(k+2) + 3c(k+1) + 4c(k) = r(k+1) - r(k)$. Determine the z-transfer function. Also obtain the waiting sequence.

Derivation of Z-transfer Function from Discrete-Time State Model:

Discrete time state model of a LDS is given by

$$X(K+1) = A\,X(K) + B\,U(K) \quad \text{--- (1)}$$

$$Y(K) = C\,X(K) + D\,U(K) \quad \text{--- (2)}$$

Taking Z-transform of eqn-(1)

$$Z\,X(Z) - Z\,X(0) = A\,X(Z) + B\,U(Z)$$

$$Z\,X(Z) - A\,X(Z) = Z\,X(0) + B\,U(Z)$$

$$[ZI - A]\,X(Z) = Z\,X(0) + B\,U(Z)$$

$$X(Z) = [ZI - A]^{-1} Z\,X(0) + [ZI - A^{-1}]\,B\,U(Z) \quad \text{--- (3)}$$

Taking Z-transform of eqn-(2)

$$Y(Z) = C\,X(Z) + D\,U(Z)$$

$$Y(Z) = C\left[[ZI - A]^{-1} Z\,X(0) + [ZI - A^{-1}]\,B\,U(Z)\right] + D\,U(Z)$$

Assuming zero initial condn

$$Y(Z) = C\,[ZI - A]^{-1} B\,U(Z) + D\,U(Z)$$

$$\boxed{\frac{Y(Z)}{U(Z)} = C\,[ZI - A]^{-1} B + D}$$

$$\frac{Y(Z)}{U(Z)} = C\,\frac{\operatorname{adj}(ZI - A)}{|ZI - A|}\,B + D$$

So the c.e. of the system will be

$$|ZI - A| = 0$$

Solution of LDS by Z-transform method

For a homogenous system, $u(k) = 0$

$$X(K+1) = A\,X(K)$$

Taking Z-transform on both side,

$$Z\,X(Z) - Z\,X(0) = A\,X(Z)$$

$$X(Z) = [ZI - A]^{-1} Z\,X(0).$$

$$x(k) = \mathcal{Z}^{-1}[X(z)] = \mathcal{Z}^{-1}[(z[zI-A]^{-1})]x(0) = \phi(k)\,x(0)$$

where $\phi(k) = \mathcal{Z}^{-1}[(z\,[zI-A]^{-1}]$

Q) Determine the discrete STM for the following system.

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(k)$$

Solⁿ: $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$|zI-A| = \left| \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right| = \begin{vmatrix} z & -1 \\ 2 & z+3 \end{vmatrix}$$

$$|zI-A| = z^2+3z+2 = (z+1)(z+2)$$

$$z[zI-A]^{-1} = \begin{bmatrix} \left(\frac{2z}{z+1} + \frac{-z}{z+2}\right) & \left(\frac{z}{z+1} + \frac{-z}{z+2}\right) \\ \left(\frac{-2z}{z+1} + \frac{2z}{z+2}\right) & \left(\frac{-z}{z+1} + \frac{2z}{z+2}\right) \end{bmatrix}$$

$$\phi(k) = \mathcal{Z}^{-1}[z(zI-A)^{-1}]$$

$$\phi(k) = \begin{bmatrix} 2(-1)^k - (-2)^k & (-1)^k - (-2)^k \\ -2(-1)^k + 2(-2)^k & -(-1)^k + 2(-2)^k \end{bmatrix}$$

<u>Solution of state eqⁿ by Recursive Method</u>

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

When given initial condition $x(0)$ and input $u(k)$ for $k=0,1,2,3\ldots$ are known then,

$k=0$, $x(1) = Ax(0) + Bu(0)$

$k=1$, $x(2) = Ax(1) + Bu(1)$

$$= A(Ax(0) + Bu(0)) + Bu(1)$$
$$= A^2x(0) + ABu(0) + Bu(1)$$

$k=2$, $x(3) = Ax(2) + Bu(2)$

$$= A[A^2x(0) + ABu(0) + Bu(1)] + Bu(2)$$

$$x(3) = A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2)$$

So in general,

$$x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{(k-1-j)} B u(j)$$

where $A^k = \phi(k)$

$A^{(k-1-j)} = \phi(k-1-j)$

Z - Transform method for solving Difference Equation.

Z- Transform can be effectively applied to obtain the solution of difference equation.

(Q) Solve the difference eqⁿ using Z-transform.

$$x(k+2) + 3x(k+1) + 2x(k) = 0, \quad x(0)=0, \quad x(1)=1$$

$$Z[x(k+2)] = z^2 X(z) - z^2 x(0) - z x(1)$$

$$Z[x(k+1)] = z X(z) - z X(0)$$

$$Z[x(k)] = X(z)$$

$$Z[x(k+2) + 3x(k+1) + 2x(k)] = Z(0)$$

$$z^2 X(z) - z^2 X(0) - z X(1) + 3z X(z) - 3z X(0) + 2X(z) = 0$$

$$[z^2 + 3z + 2] X(z) = z^2(0) + z(1) + 3z(0)$$

$$X(z) = \frac{z}{z^2+3z+2} = \frac{z}{(z+1)(z+2)} \Rightarrow \frac{X(z)}{z} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$\frac{X(z)}{z} = \frac{1}{z+1} - \frac{1}{z+2} \Rightarrow X(z) = \frac{z}{z+1} - \frac{z}{z+2}$$

Taking Inverse Z-Transform,

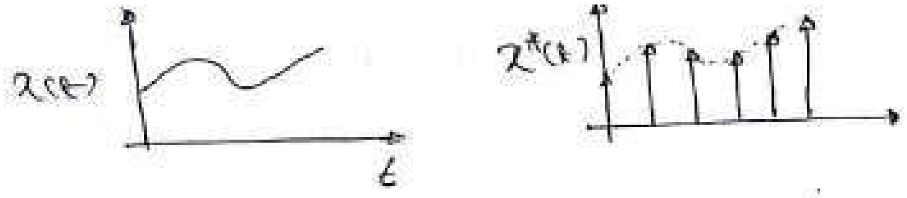
$$x(k) = (-1)^k - (-2)^k, \quad k = 0, 1, 2 \cdots$$

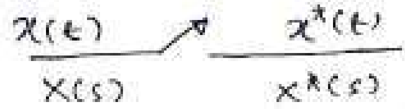
Z- Plane Analysis of Discrete-time Control System

Application of Z transform allows us to apply well understood analysis & design techniques of continuous time system to discrete time control system.

Impulse Sampling

Impulse sampler is a fictitous sampler whose o/ps are train of impulses with sampling period 'T' and strength of each ~~sample~~ impulse equal to the sampled value of the continuous time signal at the corresponding sampling instant.





The sampled signal $x^*(t)$ thus can be represented as

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\,\delta(t-kT) \quad \text{—— (1)}$$

or, $$x^*(t) = x(0)\delta(t) + x(1)\delta(t-T) + x(2)\delta(t-2T) + \cdots + x(kT)\delta(t-kT) + \cdots$$

Train of unit impulse may be defined as

$$\delta_T(t) = \sum_{k=0}^{\infty} \delta(t-kT)$$

Let's obtain Laplace transform of eq^n - (1)

$$\mathcal{L}[x^*(t)] = \mathcal{L}\left[\sum_{k=0}^{\infty} x(kT)\,\delta(t-kT)\right]$$

$$X^*(s) = x(0)\,\mathcal{L}[\delta(t)] + x(1)\,\mathcal{L}[\delta(t-T)] + x(2)\,\mathcal{L}[\delta(t-2T)] + \cdots + x(k)\,\mathcal{L}[\delta(t-kT)] + \cdots$$

$$X^*(s) = x(0) + x(1)e^{-Ts} + x(2)e^{-2Ts} + \cdots + x(k)e^{-kTs} + \cdots$$

$$= \sum_{k=0}^{\infty} x(kT)\,e^{-kTs}$$

Let's define $z = e^{sT}$, or $s = \frac{1}{T}\ln(z)$

$$X^*(s)\Big|_{s=\frac{1}{T}\ln(z)} = \sum_{k=0}^{\infty} x(kT)\,z^{-k} = X(z)$$

Hence, Laplace transform of sampled data corresponds to the z-transform of the sampled sequence for $s = \frac{1}{T}\ln(z)$.

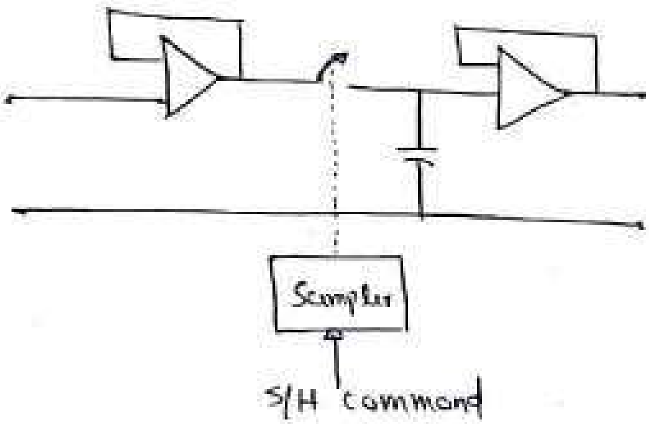
Where, z = complex variable &
T = sampling Period.

Sample & Hold Circuit

A sampler converts an analog signal into a train of amplitude modulated pulses.

The Hold ckt holds the value of the sampled pulse signal over a specified period of time.

S&H is necessary in the A/D converter to produce a no. that accurately represents the i/p signal at the sampling instant.



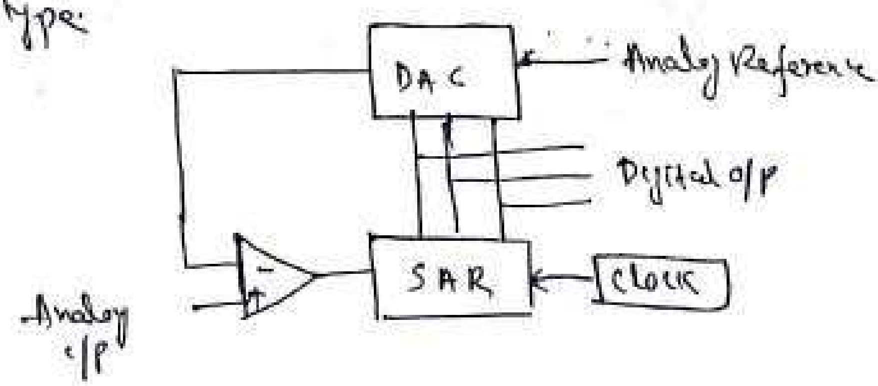
When the switch is closed, capacitor is charged to the i/p voltage. This mode is k/as tracking mode.

When the switch is off capacitor holds the voltage acquired for a specified amount of time: Hold mode.

A/D converter

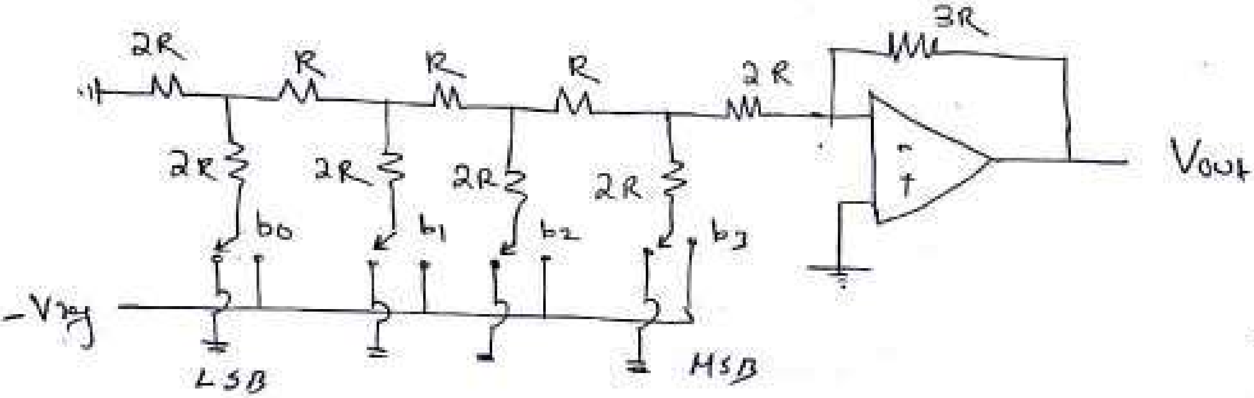
A/D converter transforms an analog signal into a digital signal or a numerically coded word.

1) Successive - Approximation type
2) Integrating type
3) Counter type
4) Parallel type



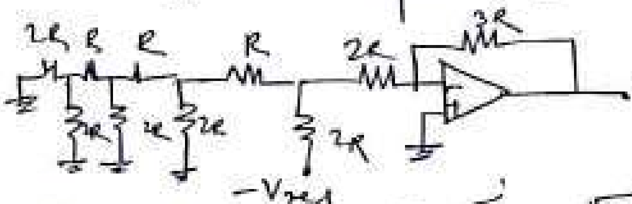
D/A Converter

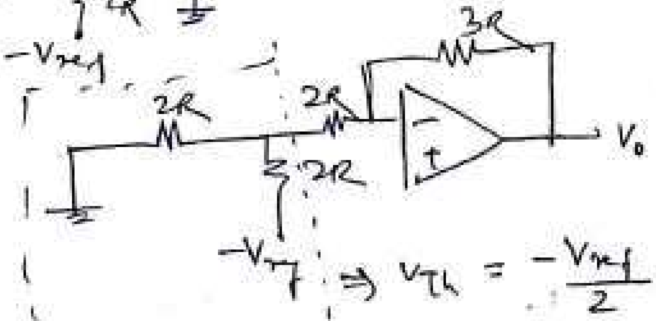
A DAC converts digital signal to its analog equivalent. The most popular ckt being R-2R Ladder ckt.



$$V_{out} = \frac{V_{b0}}{16} + \frac{V_{b1}}{8} + \frac{V_{b2}}{4} + \frac{V_{b3}}{2}$$

For bit sequence 1000, the ckt diagram will be

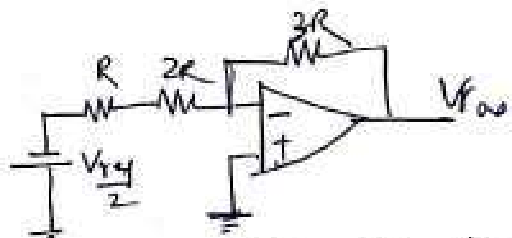




$\Rightarrow V_{Th} = -\frac{V_{ref}}{2}$

$R_{Th} \approx R$

$\Rightarrow$



$$\therefore V_{out} = \frac{V_{ref}}{2} \times \frac{3R}{3R} = \frac{V_{ref}}{2}$$

DATA HOLD CIRCUITS

Data hold circuits are employed to generate continuous time-signal $h_1(t)$ from a discrete time sequence $x(kT)$.

The signal $h(t)$ during the time interval $kT \le t < (k+n)T$ may be approximated by a polynomial in τ as follows,

$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + a_0 \quad \text{—(1)}$$

where $0 \le \tau \le T$, $h(kT) = x(kT)$

for $\tau = 0$, from eq^n-(1) $h(kT) = a_0$

Therefore, $h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + x(kT)$ —(2)

Thus, nth order hold ckt uses past $(n+1)$ discrete data to generate a signal $h(kT+\tau)$. Therefore higher order hold ckts will introduce large time delays (though provides better approximation) that risks system stability. Hence from stability point of view hold ckt that introduces minimum time delay is used.

Transfer Function of ZOH

The simplest data hold ckt that introduces minimum phase delay is obtained by substituting $n=0$ in eq^n (2).

$$h(kT+\tau) = x(kT) \quad \text{—(3)} \quad \text{for, } 0 \le t < T.$$

Thus, a ZOH works by clamping the previous sampled value until the next sample is available. Therefore ZOH is also k/as clamper ckt or staircase generator as the o/p is a staircase function.

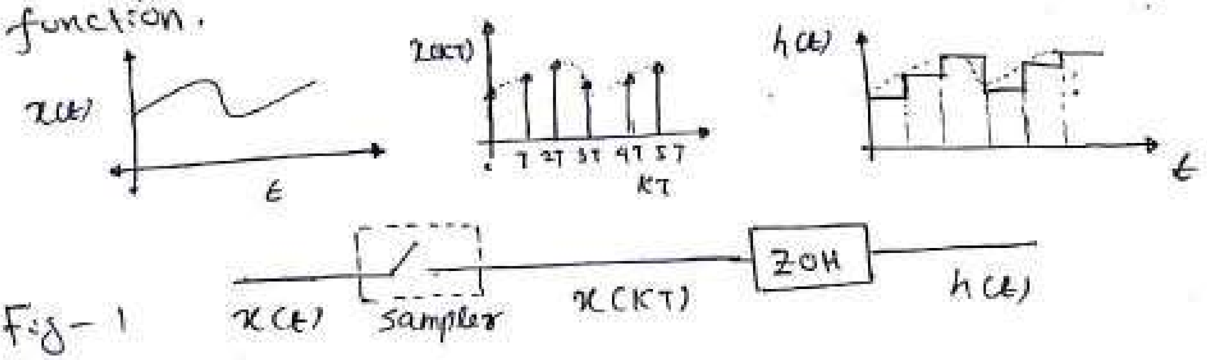
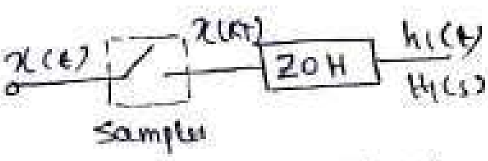


Fig-1



Real sampler & Hold ckt

Fig-2

Mathematical model of SOH ckt

Fig-3

Assume that the signal $x(t)$ is zero for $t<0$.
Then the o/p $h_1(t)$ is related to $x(t)$ as follows (from fig-2)

$$h_1(t) = x(0)[1(t) - 1(t-T)] + x(T)[1(t-T) - 1(t-2T)]$$
$$+ x(2T)[1(t-2T) - 1(t-3T)] + \cdots$$
$$= \sum_{K=0}^{\infty} x(KT)[1(t-KT) - 1(t-(K+1)T)] \quad -(4)$$

since $\mathcal{L}[1(t-KT)] = \frac{1}{s} e^{-KTs}$, Laplace Transform of eq^n (4) becomes

$$\mathcal{L}[h_1(t)] = H_1(s) = \sum_{K=0}^{\infty} x(KT)\left[\frac{e^{-KTs}}{s} - \frac{e^{-(K+1)Ts}}{s}\right]$$
$$H_1(s) = \frac{1-e^{-Ts}}{s} \sum_{K=0}^{\infty} x(KT) e^{-KTs} \quad -(5)$$

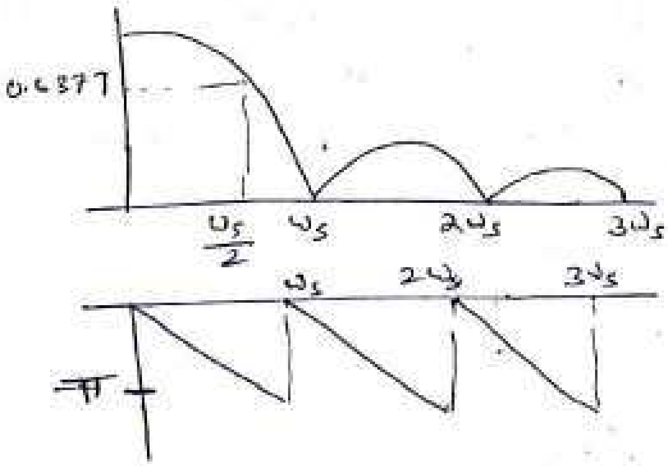
From fig-3,

$$H_2(s) = G_{ho}(s) X^*(s) \quad -(6) = G_{ho}(s) \sum_{K=0}^{\infty} x(KT) e^{-KTs}$$

But $H_1(s)$ must be equal to $H_2(s)$

$$\boxed{G_{ho}(s) = \frac{1-e^{-Ts}}{s}}$$

$$G_{Ho}(j\omega) = T \frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}} \cdot e^{-\frac{1}{2}Tj\omega}$$



$$|G_{Ho}(j\omega)| = T \left| \frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}} \right|$$

$$\angle G_{Ho}(j\omega) = \sin \frac{\omega T}{2} - \frac{\omega T}{2}$$

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Obtaining Z-Transform of functions involving the terms $(1-e^{-Ts})/s$

$$X(s) = \frac{1-e^{-Ts}}{s} G(s) = (1-e^{-Ts}) G_1(s) \quad -(1)$$

Where, $G_1(s) = \frac{G(s)}{s}$

Let $X_1(s) = e^{-Ts} G_1(s) \quad -(2)$

$$x_1(t) = \int_0^t g_0(t-\tau) g_1(\tau)\, d\tau$$

where, $g_0(t) = \mathcal{L}^{-1}[e^{-Ts}] = \delta(t-T)$

$g_1(t) = \mathcal{L}^{-1}[G_1(s)]$

Thus, $x_1(t) = \int_0^t \delta(t-T-\tau) g_1(\tau)\, d\tau = g_1(t-T)$

Hence, by writing, $Z[g_1(t)] = G_1(z)$

Z-transform of $x_1(t)$ becomes,

$$Z[x_1(t)] = Z[g_1(t-T)] = z^{-1} G_1(z)$$

From eq^n -(1) & (2)

$$X(z) = Z[G_1(s) - e^{-Ts} G_1(s)] = Z[g_1(t) - Z[x_1(t)]]$$

$$= G_1(z) - z^{-1} G_1(z) = (1-z^{-1}) G_1(z)$$

$$\boxed{X(z) = Z[X(s)] = (1-z^{-1}) Z\left[\frac{G(s)}{s}\right]}$$

(Q) Obtain the Z-Transform of $X(s) = \frac{1-e^{-Ts}}{s} \times \frac{1}{s+1}$

$$X(z) = Z\left[\frac{1-e^{-Ts}}{s} \times \frac{1}{s+1}\right] = (1-z^{-1})\, Z\left[\frac{1}{s(s+1)}\right]$$

$$= (1-z^{-1})\, Z\left[\frac{1}{s} - \frac{1}{s+1}\right] = (1-z^{-1})\, Z\left[(1-e^{-t})u(t)\right]$$

$$= (1-z^{-1})\left[\frac{z}{z-1} - \frac{ze^{T}}{ze^{T}-1}\right]$$

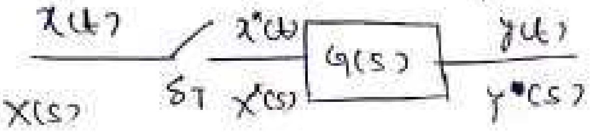
NOTE: $Z(x(kT)) = X(z)$

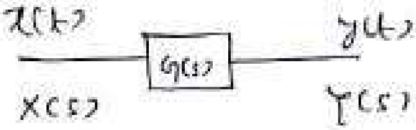
Then $Z[e^{-akT} x(kT)] = X(ze^{aT})$

Pulse Transfer Function

Impulse sampler as the i/p to the system

(1)





From the block diagram,

$Y(s) = X^*(s)G(s)$

Taking starred Laplace Transform,

$Y^*(s) = X^*(s)G^*(s)$

Or, in terms of z-transform,

$Y(z) = G(z)X(z)$

or, $\dfrac{Y(z)}{X(z)} = G(z)$

NOTE : $GX(z) \neq G(z)X(z)$

From the block diagram

$Y(s) = G(s)X(s)$

Taking starred Laplace Transform

$Y^*(s) = [G(s)X(s)]^*$

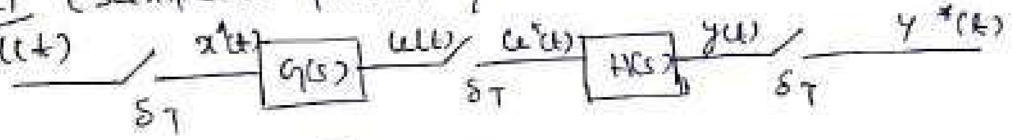
$Y^*(s) = [GX(s)]^*$

or, in terms of z-transform

$Y(z) = GX(z)$

(2) Cascaded Elements

case-1 (sampler is present b/w two cascaded blocks)



From the block diagram

$U(s) = G(s)X^*(s) \Rightarrow U^*(s) = G^*(s)X^*(s)$

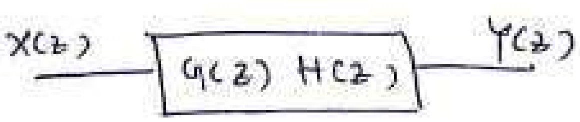
$Y(s) = H(s)U^*(s) \Rightarrow Y^*(s) = H^*(s)U^*(s)$

$\Rightarrow Y^*(s) = H^*(s)G^*(s)X^*(s)$

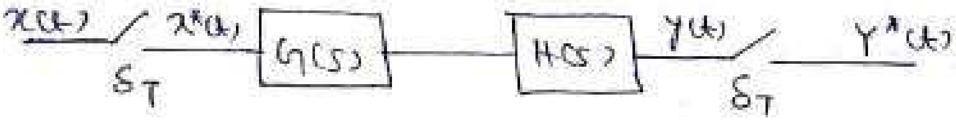
In terms of z-transform,

$Y(z) = H(z)G(z)X(z)$

or, $\boxed{\dfrac{Y(z)}{X(z)} = G(z)H(z)}$



case-2 (When sampler is absent b/w two cascaded blocks)



From the block diagram,

$$Y(s) = G(s)H(s)X^*(s) = GH(s)X^*(s)$$

$$Y^*(s) = GH^*(s)\,X^*(s)$$

In terms of z-transform,

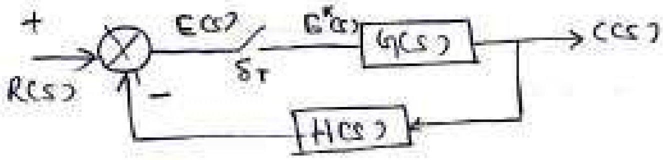
$$Y(z) = GH(z)\,X(z)$$

or, $$\boxed{\frac{Y(z)}{X(z)} = GH(z)}$$

(3) Pulse TF of closed loop system

Case-1: Error signal sampling

From the block diagram



$$C(s) = G(s)E^*(s)$$

$$E(s) = R(s) - H(s)C(s)$$

or, $$E(s) = R(s) - H(s)\left[G(s)E^*(s)\right]$$

Taking starred Laplace Transform

$$E^*(s) = R^*(s) - E^*(s)GH^*(s)$$

$$E^*(s)\left[1 + GH^*(s)\right] = R^*(s)$$

or, $$E^*(s) = \frac{R^*(s)}{1+GH^*(s)}$$

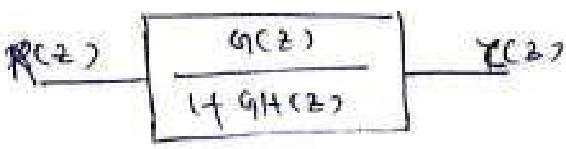
But $$C^*(s) = G^*(s)E^*(s)$$

$$C^*(s) = G^*(s)\left[\frac{R^*(s)}{1+GH^*(s)}\right]$$

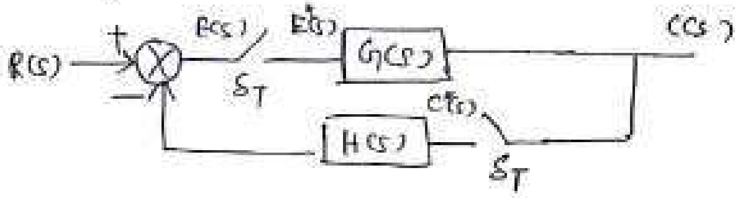
or, $$\frac{C^*(s)}{R^*(s)} = \frac{G^*(s)}{1+GH^*(s)}$$

In z-transform Notation

$$\boxed{\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}}$$



Case - 2 : Error & Feedback Sampling



From block diagram,

$$C(s) = E^*(s) G(s)$$

& $$E(s) = R(s) - H(s) C^*(s)$$

Taking Laplace Transform of above two equations,

$$C^*(s) = E^*(s) G^*(s)$$

& $$E^*(s) = R^*(s) - H^*(s) C^*(s)$$

or, $$E^*(s) = R^*(s) - H^*(s)[E^*(s) G^*(s)]$$

or, $$E^*(s)[1 + G^*(s) H^*(s)] = R^*(s)$$

or, $$E^*(s) = \frac{R^*(s)}{1 + G^*(s) H^*(s)}$$

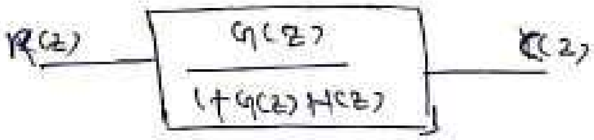
But $$C^*(s) = E^*(s) G^*(s)$$

or $$C^*(s) = \frac{R^*(s) G^*(s)}{1 + G^*(s) H^*(s)}$$

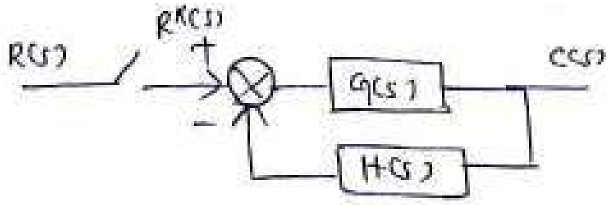
or, $$\frac{C^*(s)}{R^*(s)} = \frac{G^*(s)}{1 + G^*(s) H^*(s)}$$

In Z-transform notation,

$$\boxed{\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z) H(z)}}$$



Case - 3 : Input Sampling.



From the block diagram

$$C(s) = [R^*(s) - H(s) C(s)] G(s)$$

$$C(s) = R^*(s) G(s) - H(s) C(s) G(s)$$

or, $$[1 + G(s) H(s)] C(s) = R^*(s) G(s)$$

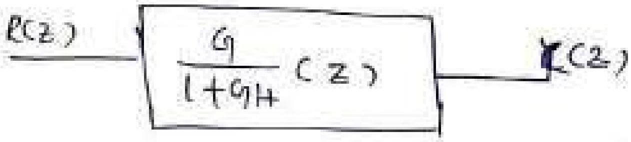
or $$C(s) = \frac{G(s)}{1 + G(s) H(s)} R^*(s)$$

or, $C^*(s) = \frac{G}{1+GH}^*(s)\, R^*(s)$

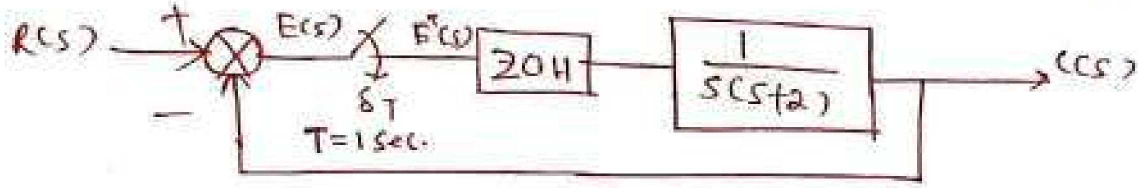
or, $\frac{C^*(s)}{R^*(s)} = \frac{G}{1+GH}^*(s)$

In Z-transform notation,

$$\frac{C(z)}{R(z)} = \frac{G}{1+GH}(z)$$



Q) Obtain the Pulse Transfer Function for the following system



Soln: Here $G(s) = \frac{1-e^{-sT}}{s} \times \frac{1}{s(s+2)}$

$H(s) = 1$

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

where $G(z) = Z[G(s)]$

$GH(z) = Z[G(s)H(s)]$

$$G(s) = 1-e^{-sT} \times \frac{1}{s^2(s+2)}$$

Let $\frac{1}{s^2(s+2)} = \frac{k_1}{s^2} + \frac{k_2}{s} + \frac{k_3}{s+2}$

$k_1 = \frac{1}{2}$, $k_2 = -\frac{1}{4}$ & $k_3 = \frac{1}{4}$

$$G(s) = (1-e^{-sT})\left[\frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{1}{s+2}\right]$$

Taking z-transform

$$G(z) = (1-z^{-1})\left[\frac{1}{2}\frac{zT}{(z-1)^2} - \frac{1}{4}\frac{z}{(z-1)} + \frac{1}{4}\frac{z}{z-e^{-2T}}\right]$$

For $T = 1$ sec,

$$G(z) = (1-z^{-1})\left[\frac{1}{2}\frac{z\times 1}{(z-1)^2} - \frac{1}{4}\frac{z}{(z-1)} + \frac{1}{4}\frac{z}{z-e^{-2\times 1}}\right]$$

$$= \frac{z-1}{z}\left[\frac{1}{2}\cdot\frac{z}{(z-1)^2} - \frac{1}{4}\cdot\frac{z}{z-1} + \frac{1}{4}\,\frac{z}{z-0.135}\right]$$

$$= (z-1)\left[\frac{2(z-0.135)-(z-1)(z-0.135)+(z-1)^2}{4(z-1)^2(z-0.135)}\right]$$

$$= \frac{0.283z+0.148}{z^2-1.135z+0.135}$$

$$GH(s) = \frac{1-e^{-sT}}{s}\cdot\frac{1}{s(s+2)}\times 1 = (1-e^{-sT})\,\frac{1}{s^2(s+2)}$$

Taking z-transform,

$$GH(z) = \frac{0.283z+0.148}{z^2-1.135z+0.135}$$

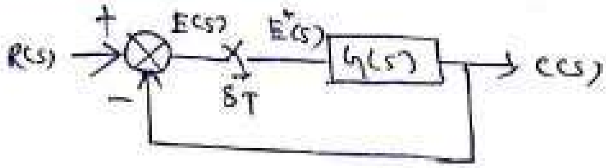
Pulse TF is given by

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)} = \frac{\dfrac{0.283z+0.148}{z^2-1.135z+0.135}}{1+\dfrac{0.283z+0.148}{z^2-1.135z+0.135}}$$

$$\frac{C(z)}{R(z)} = \frac{0.282z+0.148}{z^2-0.852z+0.283}$$

Stability Analysis of Sampled Data Control System

Overall TF of the sampled data c/s is given by,

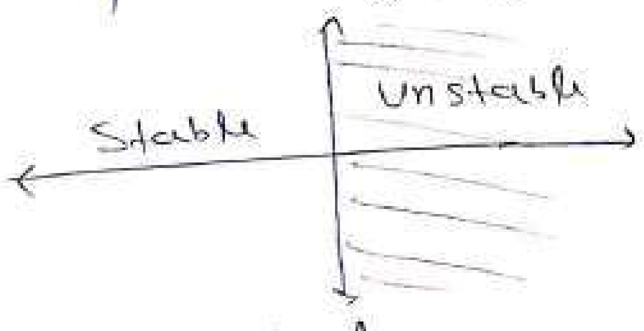


$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+GH^*(s)}$$

The characteristic of the system is $1+GH^*(s) = 0$

For the system to be stable, it requires all the roots of the above characteristic eq^n must be present in the left half of s-plane.

The region of stability in s-plane may be depicted as



s-plane

The overall TF of the above system in z-domain,

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

The characteristic eq^n will be $1+GH(z)=0$.
The stability analysis in z-domain can be carried out by mapping s-plane into z-plane.

Mapping from s-plane to z-plane

Mapping function from s-plane to z-plane is

$$z = e^{sT} \quad \text{--- (1)}$$

In s-plane imaginary axis separates the region of stability & in-stability. The region to the left of imaginary axis represents stable region and that to the right side of imaginary axis represents zone of in-stability.

(i) Mapping of imaginary axis of s-plane to z-plane.

On imaginary axis $s = \pm j\omega$

$$z = e^{sT} = e^{\pm j\omega T}, \quad |z| = |\cos\omega T \pm j\sin\omega T|$$

$$z = |z|\angle z = 1\angle \pm\omega T$$

Hence imaginary axis of s-plane mapps into an unit circle in z-plane.

(ii) Mapping of LHS of $j\omega$-axis

For any point of LHS of $j\omega$-axis in s-plane

$$s = -\alpha \pm j\omega$$

$$z = e^{(-\alpha \pm j\omega)T} = e^{-\alpha T}[\cos\omega T \pm j\sin\omega T]$$

$$|z| = e^{-\alpha T}$$

$$\angle z = \pm \omega T$$

$|z| < 1$ since T is (+)ve & α is any real number.

Hence LHS of s-plane is mapped as inside of unit circle in z-plane.

(iii) Mapping of RHS of s-plane

$$s = +\alpha \pm j\omega T$$

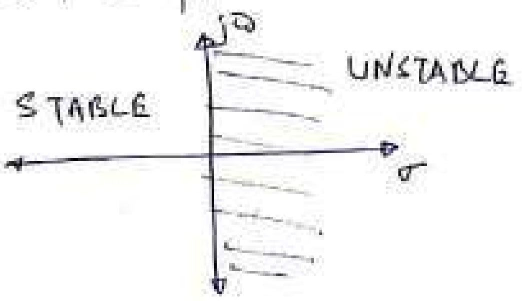
$$z = e^{(\alpha \pm j\omega)T} = e^{\alpha T}[\cos\omega T \pm j\sin\omega T]$$

$$|z| = e^{\alpha T}$$

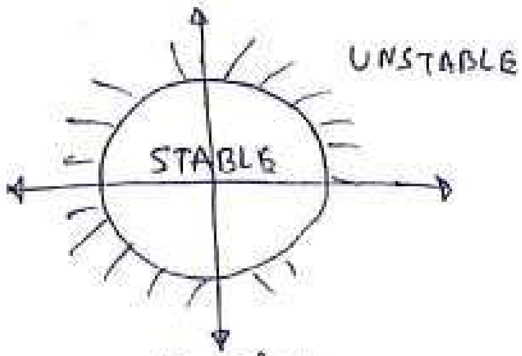
$$\angle z = \pm \omega T$$

$|z| > 1$ for any real number α and T being (+)ve.

Hence, RHS of s-plane is mapped as outside of unit circle in z-plane.



s-plane



z-plane.

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Jury's Stability Test

Let a characteristic polynomial be represented as,

$$F_1(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 = 0 \quad , \quad a_n > 0 .$$

Necessary conditions for stability are

$$F_1(1) > 0 \quad \& \quad (-1)^n F_1(-1) > 0$$

sufficient condn can be established by forming the table of co-efficients. from the table

$$|a_0| < |a_n|$$
$$|b_0| > |b_{n-1}|$$
$$\vdots$$
$$|c_0| > |c_{n-2}|$$

NOTE: Jury's array consists of $(2n-3)$ no. of rows.

Q) check the stability of the following characteristic polynomial by jury's stability criterion.

$$F(z) = 2z^4 + 7z^3 + 10z^2 + 4z + 1$$

$$F(1) = 2(1)^4 + 7(1)^3 + 10(1)^2 + 4(1) + 1 = 24 > 0$$

$$(-1)^4 F(-1) = (-1)^4 \left[2(-1)^4 + 7(-1)^3 + 10(-1)^2 + 4(-1) + 1\right]$$
$$= \left[2 - 7 + 10 - 4 + 1\right] = 2 > 0$$

		z^0	z^1	z^2	z^3	z^4
a	1	1	4	10	7	2
	2	2	7	10	4	1
b	3	$1\times1-2\times2$	$1\times4-2\times7$	$1\times10-2\times10$	$1\times7-2\times4$	
		-3	-10	-10	-1	
	4	-1	-10	-10	-3	
c	5	$(-3)\times(-3)-(-1)(-1)$	$(-3)(-10)-(-1)(-10)$	$(-3)(-10)-(-1)(-10)$		
		8	20	20		

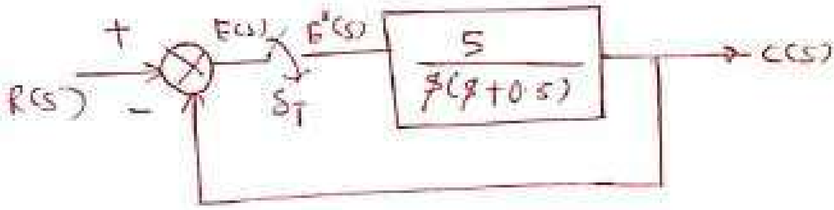
$|a_0| < |a_n| \Rightarrow |a_0| < |a_4| \Rightarrow |1| < |2|$ satisfied

$|b_0| > |b_{n-1}| \Rightarrow |b_0| > |b_3| \Rightarrow |-3| > |-1| = 3 > 1$ satisfied

$|c_0| > |c_{n-2}| \Rightarrow |c_0| > |c_2| \Rightarrow |8| > |20| \Rightarrow$ Not satisfied

Therefore the system is unstable.

(Q) Determine the pulse TF and stability of the sampled data control system shown below for sampling time (a) $T = 0.5$ & (b) $T = 1$ sec.



Sol^n: $G(s) = \dfrac{5}{s(s+0.5)} = \dfrac{k_1}{s} + \dfrac{k_2}{s+0.5}$

$$k_1 = \frac{5}{\cancel{s}(s+0.5)} \times \cancel{s} \Big|_{s=0} = \frac{5}{0.5} = 10$$

$$k_2 = \frac{5}{s\cancel{(s+0.5)}} \times \cancel{(s+0.5)} \Big|_{s=-0.5} = -10$$

$$G(s) = \frac{10}{s} + \frac{-10}{s+0.5}$$

Taking, z-Transform,

$$G(z) = 10\left[\frac{z}{z-1} - \frac{z}{z-e^{-0.5T}}\right] = \frac{10z(1-e^{-0.5T})}{z^2 - z(1+e^{-0.5T}) + e^{-0.5T}}$$

Here, $G(s) = \dfrac{5}{s(s+0.5)}$ & $H(s) = 1$

So, $G(s)H(s) = GH(s) = \dfrac{5}{s\times(s+0.5)} \times 1$

$$GH(z) = \frac{10z(1-e^{-0.5T})}{z^2 - z(1+e^{-0.5T}) + e^{-0.5T}}$$

The Pulse TF of the overall system is given by,

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)} = \frac{\dfrac{10z(1-e^{-0.5T})}{z^2 - z(1+e^{-0.5T}) + e^{-0.5T}}}{1 + \dfrac{10z(1-e^{-0.5T})}{z^2 - z(1+e^{-0.5T}) + e^{-0.5T}}}$$

or, $$\frac{C(z)}{R(z)} = \frac{10z(1-e^{-0.5T})}{z^2 - z(11e^{-0.5T} - 9) + e^{-0.5T}}$$

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The c.e. is $z^2 - z(11e^{-0.5T} - 9) + e^{-0.5T} = 0$

(a) For $T = 0.5$ sec, c.e. becomes

$$z^2 - z(11e^{-0.5\times0.5} - 9) + e^{-0.5\times0.5} = 0$$

or, $z^2 - 0.42z + 0.78 = 0$

For, Billinear Transformation, put $z = \frac{1+\omega}{1-\omega}$

$$\left(\frac{1+\omega}{1-\omega}\right)^2 - 0.42\left(\frac{1+\omega}{1-\omega}\right) + 0.78 = 0$$

or, $1.36\omega^2 + 0.44\omega + 2.2 = 0$

Routh's Array of above eqn will be

$$\begin{array}{c|cc} \omega^2 & 1.36 & 2.2 \\ \omega^1 & 0.44 & 0 \\ \omega^0 & 2.2 & \end{array}$$

Since, there is no sign change in 1st column of Routh's array, hence, all the roots of the c.e. lies within the unit circle centred at origin of z-plane. So the system is stable.

(b) For $T = 1$ sec, c.e becomes

$$z^2 - z(11e^{-0.5\times1} - 9) + e^{-0.5\times1} = 0$$

or, $z^2 + 2.33z + 0.606 = 0$

Applying Billinear Transform, $z = \frac{1+\omega}{1-\omega}$

$$\left(\frac{1+\omega}{1-\omega}\right)^2 + 2.33\,\frac{1+\omega}{1-\omega} + 0.606 = 0$$

or, $-0.724\omega^2 + 0.788\omega + 3.93 = 0$

Routh's array of the above system will be,

$$\begin{array}{c|cc} \omega^2 & -0.724 & 3.93 \\ \omega^1 & 0.788 & 0 \\ \omega^0 & 3.93 & \end{array}$$

Since, there is one sign change in 1st column of Routh's array it indicates one of the roots of c.e. present outside of the unit circle in z-plane. Hence the system is unstable.