Lecture Notes

On

Advance Control Systems

7^{Th} Semester Electrical Engineering

Part 5: Modeling and Analysis of Nonlinear Control Systems

By

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Disclaimer

Course Objective:

The objective of this course is to equip students with a deep understanding of discrete-time control systems, state variable analysis, and nonlinear system behavior. Students will learn to analyze, design, and implement control strategies using modern techniques, including Z-transform methods, state-space representations, and Lyapunov stability analysis.

Course Outcome:

- **Analyze**: Analyze discrete-time and continuous-time control systems using Z-transform and state-space methods to determine system behavior.
- **Design**: Design control systems utilizing feedback strategies, pole placement, and observer design to achieve specified performance criteria.
- **Evaluate**: Evaluate the stability of linear and nonlinear systems using Routh's criterion and Lyapunov's methods, assessing their robustness.
- **Apply**: Apply techniques for modeling and simulating nonlinear systems, including phase plane and describing function methods, to solve practical engineering problems.

Non Linear Control System

Non-linear systems

→ In contrast to linear systems, non-linear systems does not posses principle of homogenity and superposition.

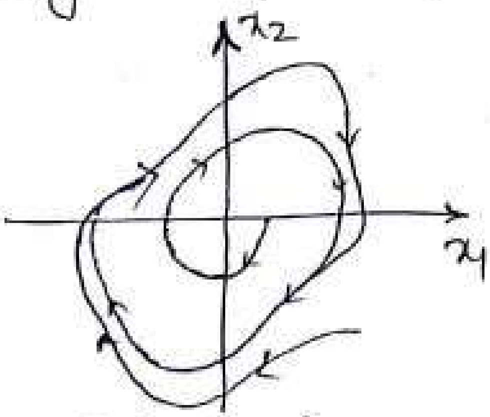
→ Further any change in i/p to the non-linear system, there will be considerable change in shape, size, frequency of the response.

→ Therefore application of sinusoidal i/p to a non-linear system may cause a o/p that will contain harmonics of the i/p aswell.

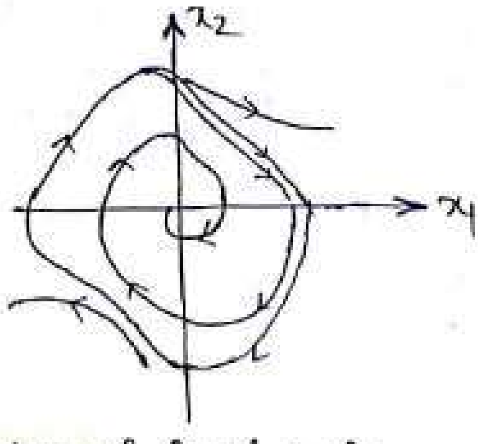
$$A \sin\omega t \longrightarrow \boxed{\text{Non-Linear s/s}} \longrightarrow \sum B_n \sin(n\omega t + \phi_n)$$

→ Besides this, non-linear systems posses some specific features like limit cycle and jump resonance.

→ The disturbed non-linear system, even when staying within its tolerance limits of steady-state oscillation, may exhibit closed trajectory. These closed trajectories are called as limit cycles. These oscillations will be of constant magnitude and frequency which may not be sinusoidal regardless of the magnitude of the input or initial conditions.



Stable limit cycle.



Unstable limit cycle.

→ Bending of frequency response curve of a non-linear system near the resonance frequency giving hysteresis like phenomenon is called as jump resonance.

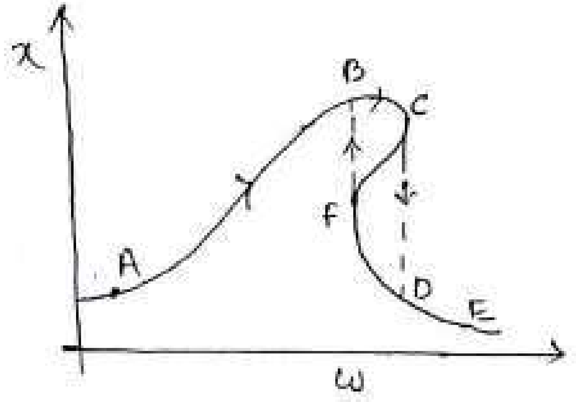


Fig-1

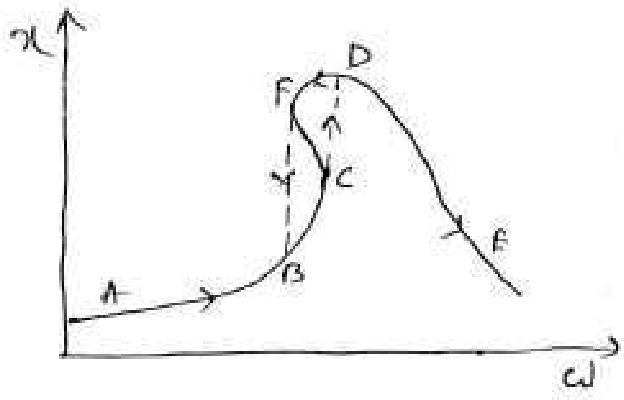


Fig-2

As the input frequency is gradually increased from zero, holding the input amplitude fixed, the measured response follows the curve through the points A, B and C, but at ~~an~~ C an increment in freq. results in a discontinuous jump from down to the point D, after which with further increase in frequency, the responce curve follows through DE. If the freq. is now decreased, the responce follows the curve EDF with a jump up to B occuring at F and then response curve moves towards A.

Common Nonlinearities

→ Non linearities can be classified as incidental and intentional.

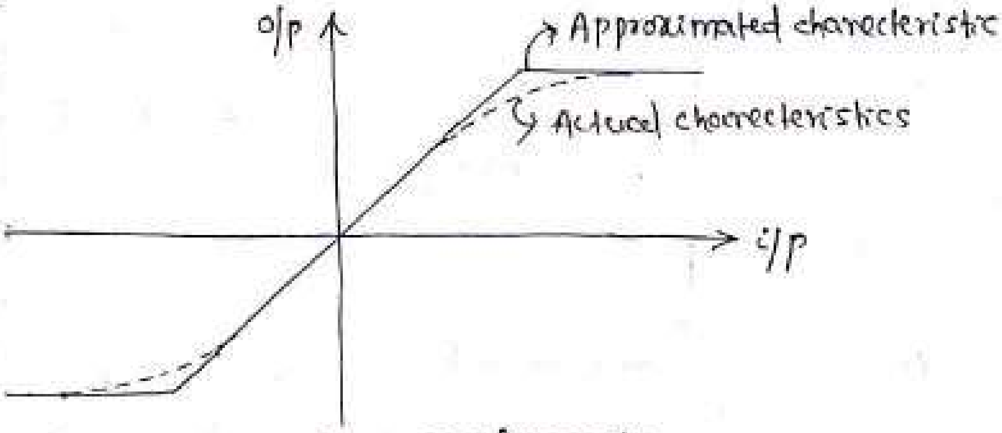
→ Incidental non linearities are those which are present inherently in the system such as saturation, dead-zone, coulomb friction, stiction, backlash etc.

→ Intentional nonlinearities are those which are deliberately introduced in the system to modify the system performance. The most common example of this type of nonlinearity is a relay.

Saturation

Most practical systems when driven by sufficiently large signals, exhibit the phenomenon of saturation due to limitation of physical capabilities of their component.

Many components such as amplifier have output proportional to the i/p in a limited range of i/p signals. When i/p exceeds the range, the o/p tends to become nearly constant.



Saturation non linearity.

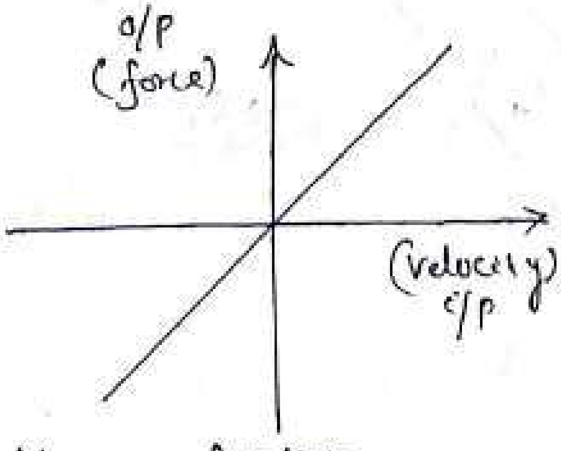
Friction

The predominant frictional force is called as viscous friction which is proportional to the relative velocity of sliding surface. i.e. viscous friction = $f\dot{x}$
where f is a constant and $\dot{x}$ is the relative velocity.

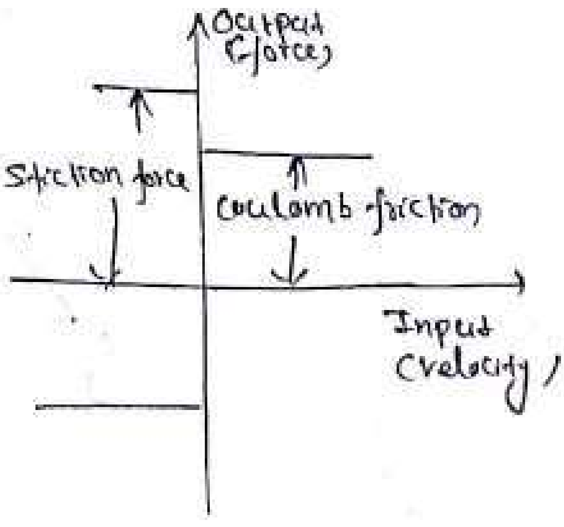
In addition to the viscous friction there exists two non linear frictions — coulomb friction & stiction.

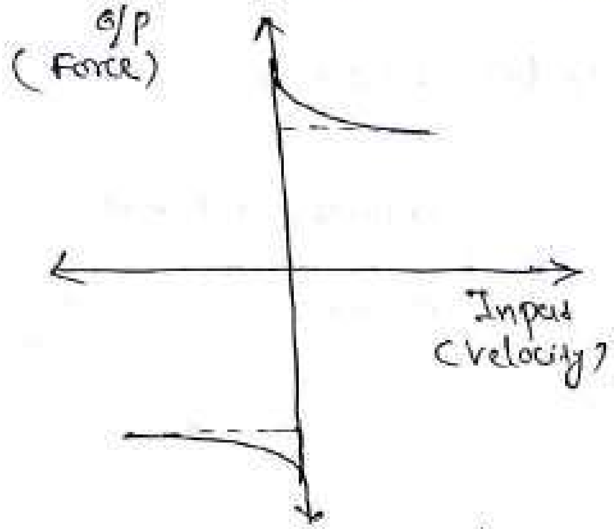
Coulomb friction is a constant retarding force always opposing the relative motion.
Stiction is the force required to initiate the motion.

The force of stiction is always greater than that of coulomb friction since due to interlocking of surface irregularities, more force is required to move an object from rest than to maintain it is motion.

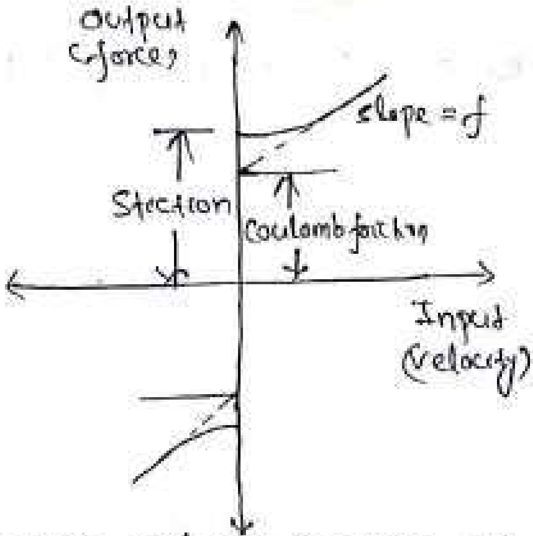


Viscous friction.





Actual stiction and coulomb friction.

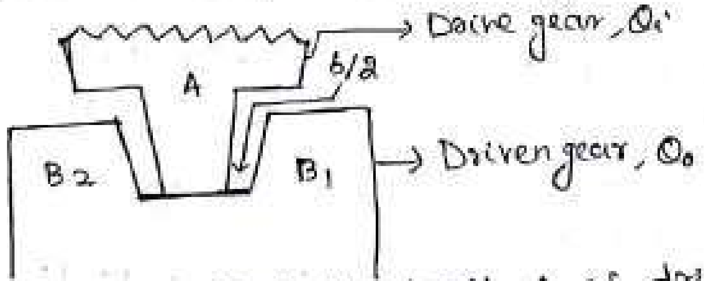


Stiction, coulomb friction and viscous friction.

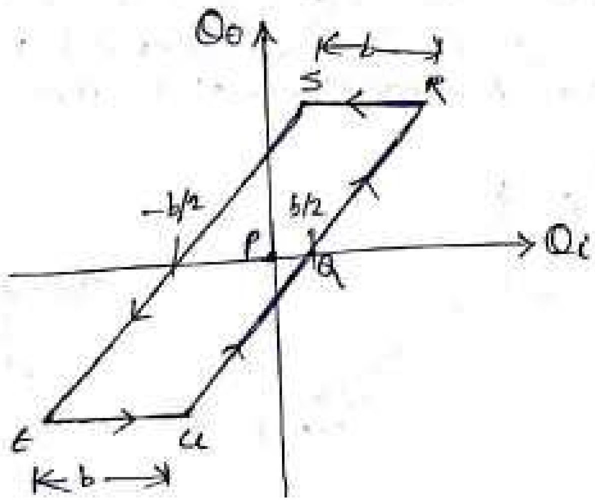
Backlash

It is a nonlinearity commonly found in mechanical transmissions systems such as gear trains an linkages.

Backlash infact is the play between the teeth of the drive gear and those of the driven gear.



Above figure shows that the tooth A of drive gear is located midway beetween the teeth B1 and B2 of the driven gear.



As tooth A is driven counter clockwise from the present position, no output motion takes place until the tooth A makes contact with the tooth B_1 of the driven gear after travelling a distance of b/2. This o/p motion corresponds to the segment pq.

After the contact is made the driven gear rotates clockwise through the same angle as the drive gear as shown by the line segment qr.

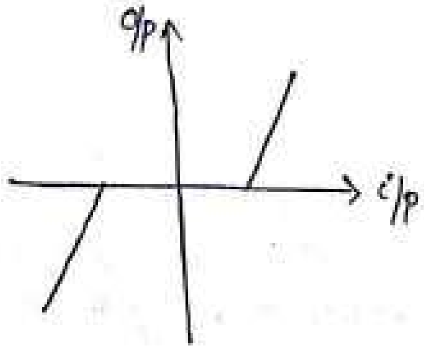
When the motion is reversed, the tooth of drive gear makes contact with tooth B_2 of driven gear after travelling a distance of b. Till this point o/p motion will be zero as shown by the line rs.

After tooth A establishes contact with tooth B_2, the driven gear now moves counterclockwise direction as depicted by the line st.

As the driven gear is reversed, the driven gear is again at stand still for the segment tu and then follows the drive gear along uq. This completes one cycle of o/p motion.

From the i/p - o/p characteristic it is seen that, for a given i/p, the o/p is multivalued. Which particular o/p will result for a given i/p depends upon the history of o/p. Hence this non-linearity is regarded as memory type nonlinearity.

Dead Zone



Sometimes for a specific range of i/p the o/p of the system will be zero. This type of non-linearity is regarded as dead zone.

This type of non-linearity is mainly exhibited by motors, actuators, dc servo motors etc.

Relay

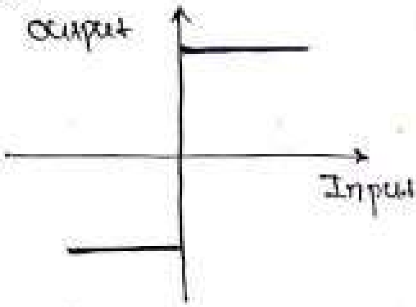
Relay is a non-linear power amplifier which can provide

Large power amplification inexpensively.

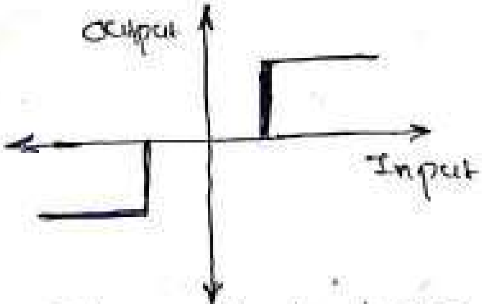
A relay exhibits two types of non-linearities — dead zone, saturation or combination of both.

The relay coil requires a finite amount of current to actuate relay. This leads to dead zone non-linearity.

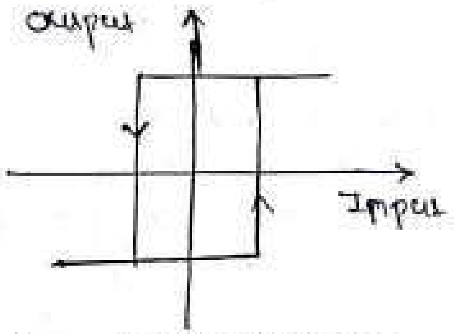
Relay exhibits hysteresis because a large coil current is needed to close the relay than the current at which relay drops out.



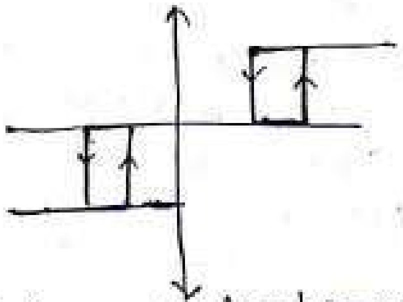
Ideal Relay.



Relay with dead zone.



Relay with hysteresis.



Relay with dead zone & hysteresis.

Stability of Non-Linear Systems

The stability criteria that is used for a linear time invariant systems are —

1) For free system: A system is stable with zero input and arbitrary initial condn if the resulting trajectory tends towards the equilibrium state.

2) For forced system: A system is stable if with bounded input, the system output is bounded.

But the above two stability notions can not be extended to non-linear systems. Because unlike linear systems non-linear systems exhibits multiple equilibrium points. So the system trajectory in phase plane may be seen moving away from one equilibrium point and as time progress may enter to other equilibrium states.

Thus for non linear systems, stability must be explained in relative to equilibrium states.

Some of the definations on stability are as follows .....

Stability: A system defined as $\dot{x} = F(x)$ is stable at the origin, if for every initial state $x(t_0)$, sufficiently close to origin, $x(t)$ remains near the origin for all t.

Asymptotic stability: A system is asymptotically stable if $x(t)$ in fact approches origin as $t \to \infty$.

Asymptotic stability In-the-large: It is asympotically stable in the large if it is asymptotically stable for every initial state regardless of how far or near it is from origin.

Stability of a non linear system can be investigated using the following four methods.

1. Phase-plane method
2. Describing function method
3. Liapunov stability Theorem
4. Popov's stability criterion

Phase Plane Method

Phase plane method is a graphical technique for solving 2nd order non-linear differential equation only.

Let's consider a 2nd order differential eqn of the form,

$$\frac{d^2x}{dt^2} + A\frac{dx}{dt} + Bx = 0 \quad \text{---(1)}$$

Let the state of the system be described by two variables x_1 and x_2. Such that $x_1 = x$ & $x_2 = \frac{dx}{dt} = \dot{x}$

Eq^n-(1) can be written as

$$\dot{x}_2 + Ax_2 + Bx_1 = 0 \Rightarrow \dot{x}_2 = -(Ax_2 + Bx_1)$$

Hence the state eq^n is represented as

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(Ax_2 + Bx_1) \end{aligned}\right\} \quad \text{— (2)}$$

So, $$\boxed{\frac{\dot{x}_2}{\dot{x}_1} = -\frac{Ax_2 + Bx_1}{x_2}} \quad \text{— (3)}$$

Phase Plane: The co-ordinate plane with axes that correspond to the dependent variable $x_1 = x$ and it's first derivative $x_2 = \dot{x}$ is called phase plane.

Phase Trajectory: The curve described by the state point (x_1, x_2) in the phase plane with time as running parameter is called a phase trajectory.

Phase Portrait: Family of phase trajectories for different initial conditions is K/as phase portrait. It gives information about stability and limit cycle.

Eq^n—(3) represents the slope of phase trajectory.

<u>Autonomous System</u>

A time invariant system can be represented as,

$$\dot{x} = f(x, u)$$

If the input u is constant, then the above eq^n can be written as

$$\dot{x} = f(x)$$

Any system represented by the above eq^n is called as autonomous system / unforced system.

Singular Point

A point on the phase at which the derivatives of all the state variables are zero is called as a singular point or equilibrium point.

If a system at singular point left unaltered, then it continues to lie in that state as the change in state represented by the derivative of the state is zero.

If the autonomous system is represented as $\dot{x} = AX$ and x_e represent equilibrium states then

$$\dot{x} = Ax_e = 0$$

$$\Rightarrow Ax_e = 0$$

If A is non-singular then $x_e = 0$ is the only sol^n.

Otherwise if all the eigenvalues of the system are non zero then origin is the only singular point.

Stability from Phase Plane

The location of roots of the characteristic eq^n in s-plane determines the characteristic of singular point. A system is said to be stable if the phase trajectory reaches the singular point.

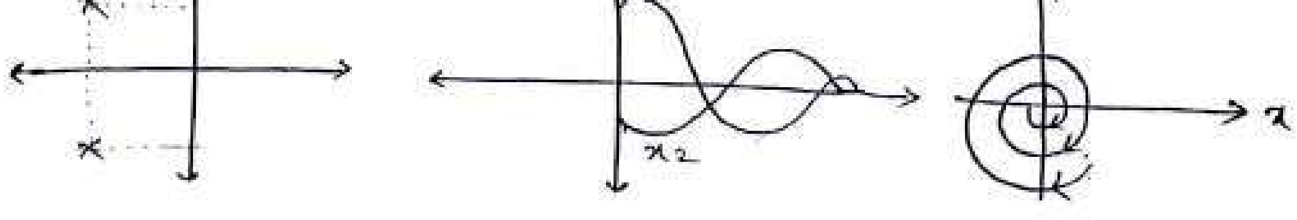
Case-1

Complex roots having -ve real part.

$\lambda_1 = -\alpha - j\beta$ and $\lambda_2 = -\alpha + j\beta$, $\alpha > 0$ & $\beta > 0$

The response of the system will be $y(t) = c_1 e^{-\alpha t} \sin(\beta t + c_2)$

For this case the phase trajectory will be a logarithmic spiral with origin as singular point which is called as stable focus.



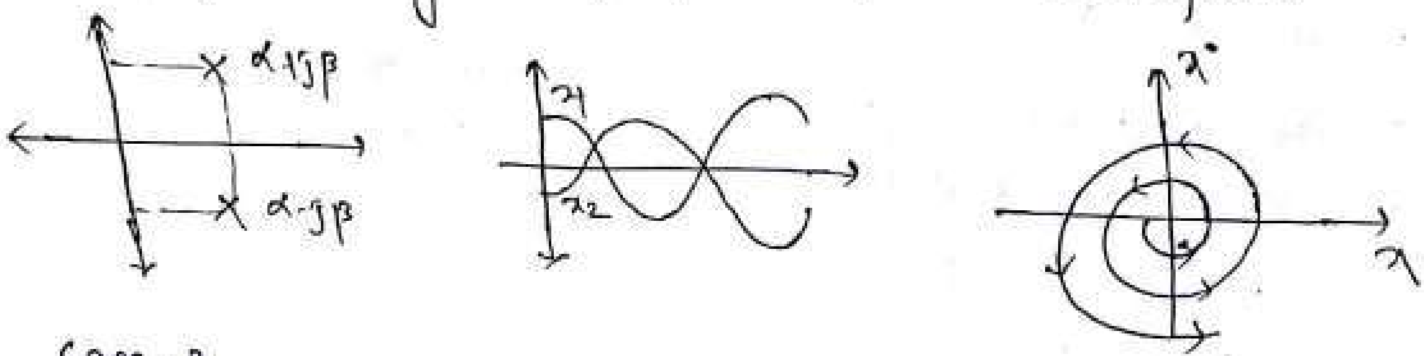
Case - 2

Complex roots having (+)ve real part.

$\lambda_1 = \alpha + j\beta$ and $\lambda_2 = \alpha - j\beta$, $\alpha > 0$ & $\beta > 0$

The response of the system will be $y(t) = c_1 e^{\alpha t} \sin(\beta t + c_2)$

The phase trajectory will be logarithmic spiral expanding out of the singular point called as unstable focus.

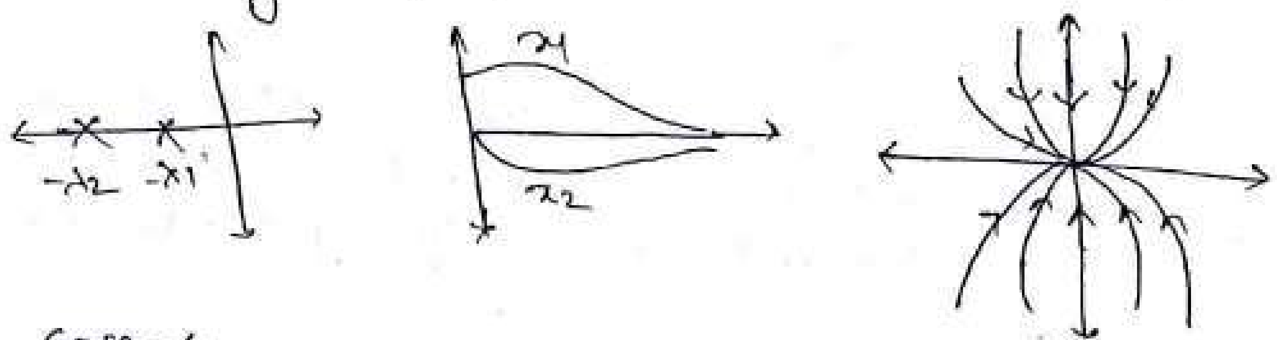


Case - 3

-ve real distinct roots. $-\lambda_1$ and $-\lambda_2$

The response of the system will be $y(t) = c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}$

The singular point is called as stable node.

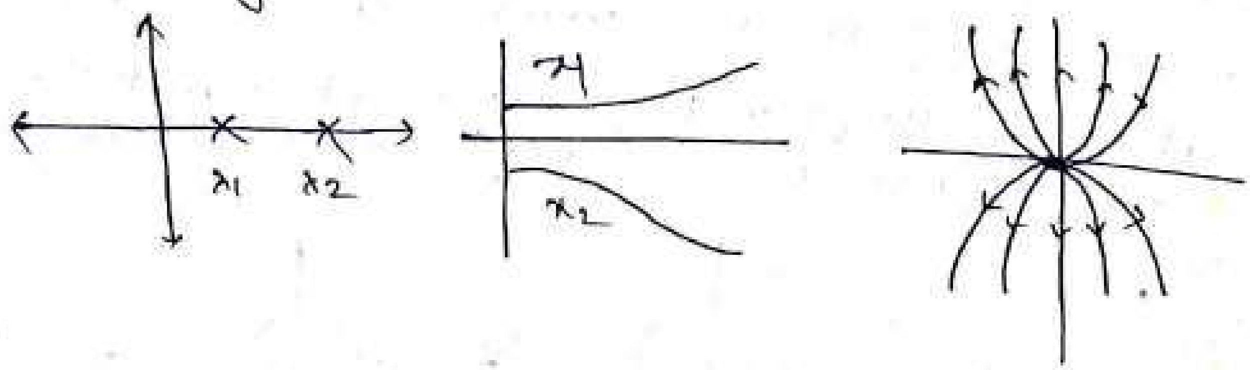


Case - 4

+ve real distinct roots, λ_1 & λ_2

The response of the system will be $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

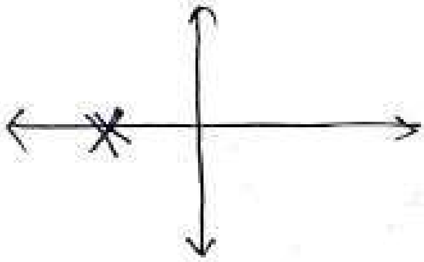
The singular point is called as unstable node.

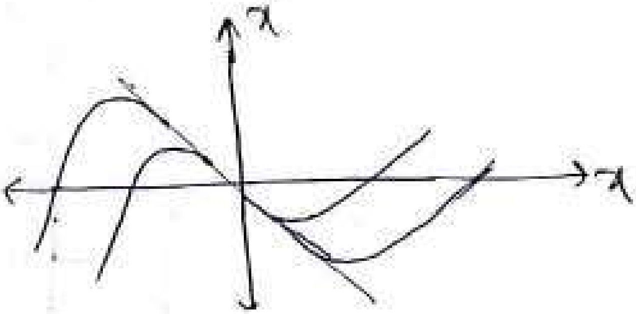


Case - 5

System with repeated roots i.e. λ_1 & λ_1

$y(t) = c_1(e^{\lambda_1 t} + t e^{\lambda_2 t})$



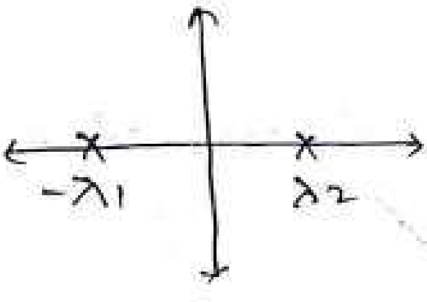


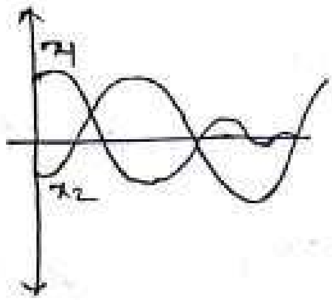
Case - 6

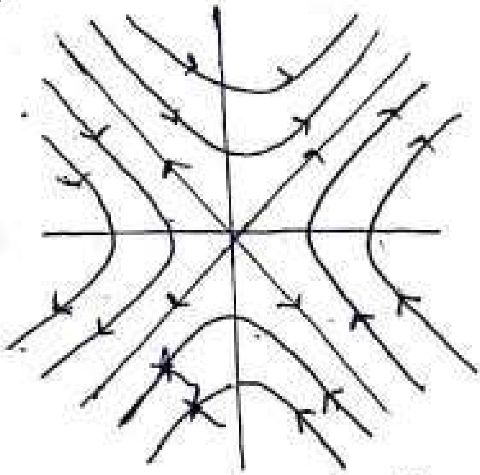
System with one (+)ive root and one (-)ve root, $-\lambda_1$ & λ_2

The response of the system will be $y(t) = c_1 e^{-\lambda_1 t} + c_2 e^{\lambda_2 t}$

The singular point for this phase-trajectory is k/a-s saddle point. Here the straight line due to the (-)ve root provides a trajectory that enter the singular point, while the straight line trajectory due to the positive root, leave the singular point.





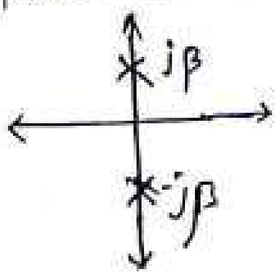


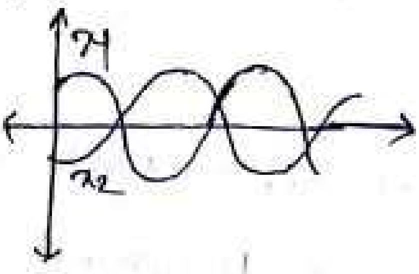
Case - 7

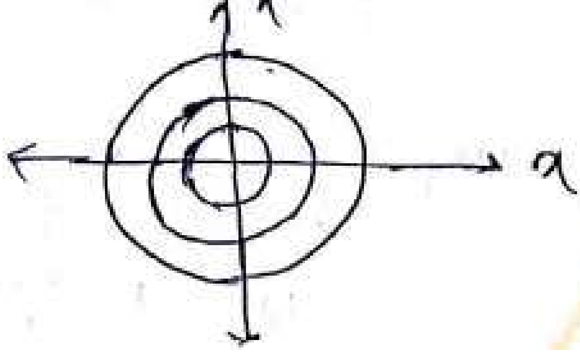
A system having complex conjugate roots with zero real part $\lambda_1 = j\beta$ and $\lambda_2 = -j\beta$.

The response of the system will be $y(t) = c_1 \sin(\beta t + c_2)$

The singular point on the phase trajectory is called as a centre or vortex.

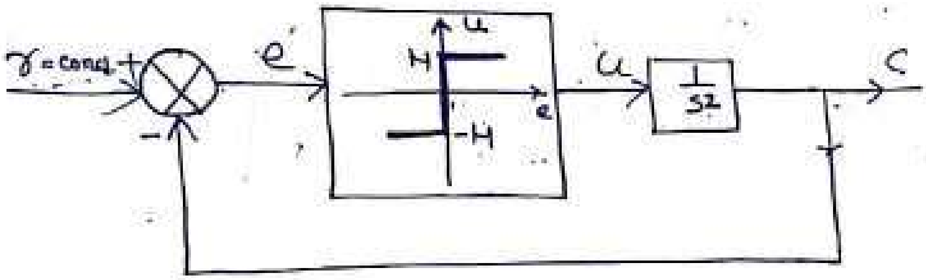






Construction of phase Trajectory by analytical method

Q) Consider an ON-OFF y/s as shown in the figure. The non-linear element being an ideal relay having charecteristic as shown in the figure. Derive the analytical expression for constructing phase portrait.



Step-1

$$e = r - c$$

$$\dot{e} = -\dot{c} \quad \left[\text{as } r = \text{const } \dot{r} = 0 \right.$$

$$\ddot{e} = -\ddot{c} \quad \text{—— (1)}$$

Step-2

$$u \times \frac{1}{s^2} = c$$

$$\Rightarrow u = c\, s^2$$

$$\Rightarrow u = \ddot{c} \quad \text{—— (2)}$$

Substituting the value of u from eq^n-(2) in eq^n-(1)

$$\boxed{\ddot{e} = -u} \quad \text{—— (3)}$$

Step-3

Choosing the state vector $x_1 = e$
$x_2 = \dot{e}$

$$\dot{x}_1 = \dot{e} = x_2 \qquad \Big| \qquad \dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{e} = -u \qquad \Big| \qquad \dot{x}_2 = -u$$

Step-4

The output of ON-OFF controller is

$$u = M \operatorname{sgn} e = M \operatorname{sgn} x_1$$

The state eqn can be expressed as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -M \operatorname{sgn} x_1$$

The slope of the eqn is

$$\frac{\dot{x}_2}{\dot{x}_1} = -\frac{M \operatorname{sgn} x_1}{x_2}$$

Separating the variable and integrating we get

$$\int x_2 \cdot \dot{x}_2 = \int -M \operatorname{sgn} x_1 \, \dot{x}_1$$

$$\Rightarrow \int_{x_2(0)}^{x_2} x_2 \, dx_2 = -M \int_{x_1(0)}^{x_1} \operatorname{sgn} x_1 \, dx_1$$

$$\Rightarrow \left[\frac{x_2^2}{2}\right]_{x_2(0)}^{x_2} = -M \left[-x_1\right]_{x_1(0)}^{x_1}$$

$$\Rightarrow \frac{x_2^2}{2} - \frac{x_2^2(0)}{2} = -M x_1(0) + M x_1$$

$$\Rightarrow \boxed{x_2^2 = 2Mx_1 - 2Mx_1(0) + x_2^2(0)}$$

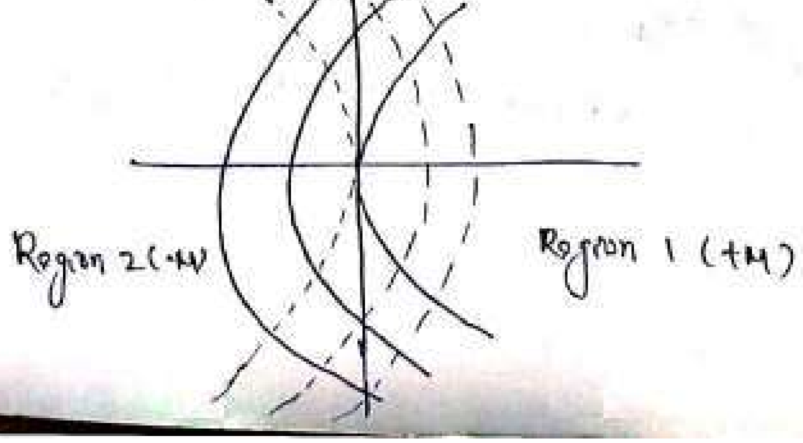
Case-1 If x_1 = e positive, relay o/p is $+M$

$$x_2^2 = 2Mx_1 - 2Mx_1(0) + x_2^2(0); \quad x_1 > 0$$

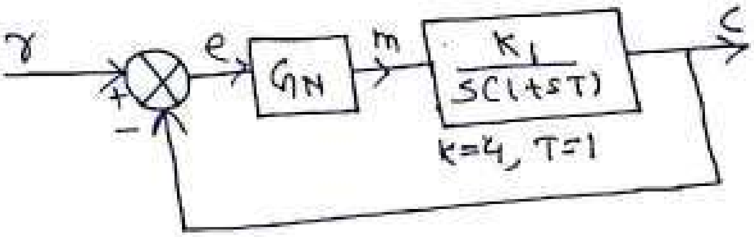
Case-2 If x_1 = e negative, relay o/p is $-M$

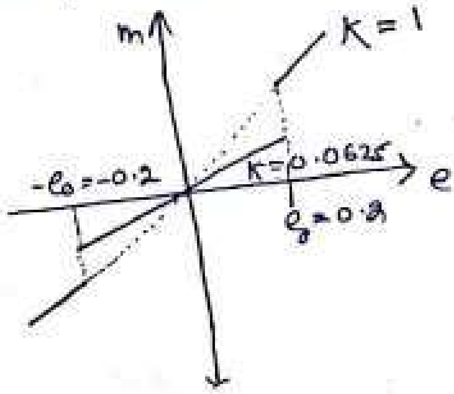
$$x_2^2 = 2Mx_1 - 2Mx_1(0) + x_2^2(0) \quad x_1 < 0$$

The phase portrait for above eqn is shown below.



Q) Consider a control system having a non-linear gain as described in the figure below. Draw the phase plane trajectory of the system by analytical method.





step-1 Non-linear block can be modelled as

$$m = e \quad \text{for } |e| > e_0 \quad \text{--- (1)}$$

$$m = Ke \quad \text{for } |e| < e_0$$

$$m = 0.0625e \quad (\text{As } K = 0.0625) \quad \text{--- (2)}$$

step-2

$$\frac{c}{m} = \frac{K_1}{S(1+ST)} = \frac{K_1}{S + S^2 T}$$

$$\Rightarrow cS^2T + cS = K_1 m$$

Taking Inverse Laplace transform

$$T\frac{d^2c}{dt^2} + \frac{dc}{dt} = K_1 m$$

$$T\ddot{c} + \dot{c} = K_1 m \quad \text{--- (3)}$$

step-3

from the block diagram $e = r - c$

$$c = r - e$$

Substituting the value of c in eq^n-(3)

$$T(\ddot{r} - \ddot{e}) + (\dot{r} - \dot{e}) = K_1 m$$

$$T\ddot{r} - T\ddot{e} + \dot{r} - \dot{e} = K_1 m$$

$$T\ddot{r} + \dot{r} = T\ddot{e} + \dot{e} + K_1 m \quad \text{--- (4)}$$

step-4

For step input $\dot{r} = 0$ and $\ddot{r} = 0$

Eq^n-(4) can be written as

$T\ddot{e} + \dot{e} + k_1 m = 0 \quad -(5)$

Step 5

for $|e| > e_0$

$T\ddot{e} + \dot{e} + k_1 e = 0 \quad$ [As $m = e$ for $|e| > e_0$

$\ddot{e} = -\frac{1}{T}\dot{e} - \frac{k_1}{T} e$

Substituting the value of $T = 1$ & $k_1 = 4$,

$\ddot{e} = -\dot{e} - 4e \quad —— (6)$

Assuming $x_1 = e$

$x_2 = \dot{e} = \dot{x}_1$

$\dot{x}_2 = -\dot{x}_1 - 4x_1 = -x_2 - 4x_1$

In state space,

$\dot{x}_1 = x_2$

$\dot{x}_2 = -4x_1 - x_2$, $\quad A = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix}$

So the eigenvalues of the system will be,

$|sI - A| = 0$

$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \right| = 0$

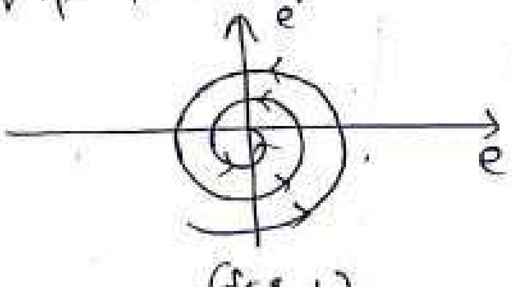
$\begin{vmatrix} s & -1 \\ 4 & s+1 \end{vmatrix} = 0 \Rightarrow s^2 + s + 4 = 0$

$s = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$

$s_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$ and $s_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{3}i$

Since $|A|$ is non-singular, origin is the only singular point or the equilibrium point for the system.

As the eigen values are complex with (-)ve real part, the singular point is a stable focus for $|e| > e_0$ (0.2)



(fig-1)

Step - 6

for $|e| < e_0$

$$m = Ke = 0.0625e$$

so eq^n-(5) becomes

$$T\ddot{e} + \dot{e} + KK_1 e = 0$$

substituting the value $T = 1$ & $K_1 = 4$

$$\ddot{e} + \dot{e} + 4 \times 0.0625 e = 0 \quad \text{--- (7)}$$

$$\ddot{e} + \dot{e} + 0.25 e = 0$$

Assuming $x_1 = e$

$$x_2 = \dot{e} = \dot{x}_1$$

$$\dot{x}_2 + x_2 + 0.25 x_1 = 0$$

$$\Rightarrow \dot{x}_2 = -0.25 x_1 - x_2$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -0.25 x_1 - x_2$$

$$A = \begin{bmatrix} 0 & 1 \\ -0.25 & -1 \end{bmatrix}$$

So the eigen value of the system will be,

$$|\lambda I - A| = 0$$

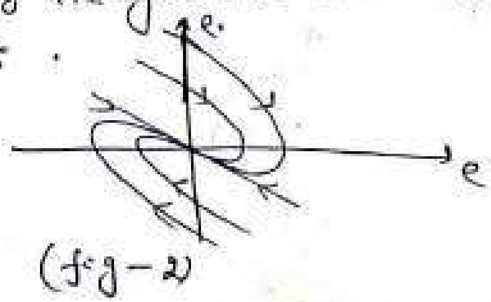
$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.25 & -1 \end{bmatrix} \right| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ 0.25 & \lambda+1 \end{vmatrix} = 0$$

$$\lambda^2 + \lambda + 0.25 = 0$$

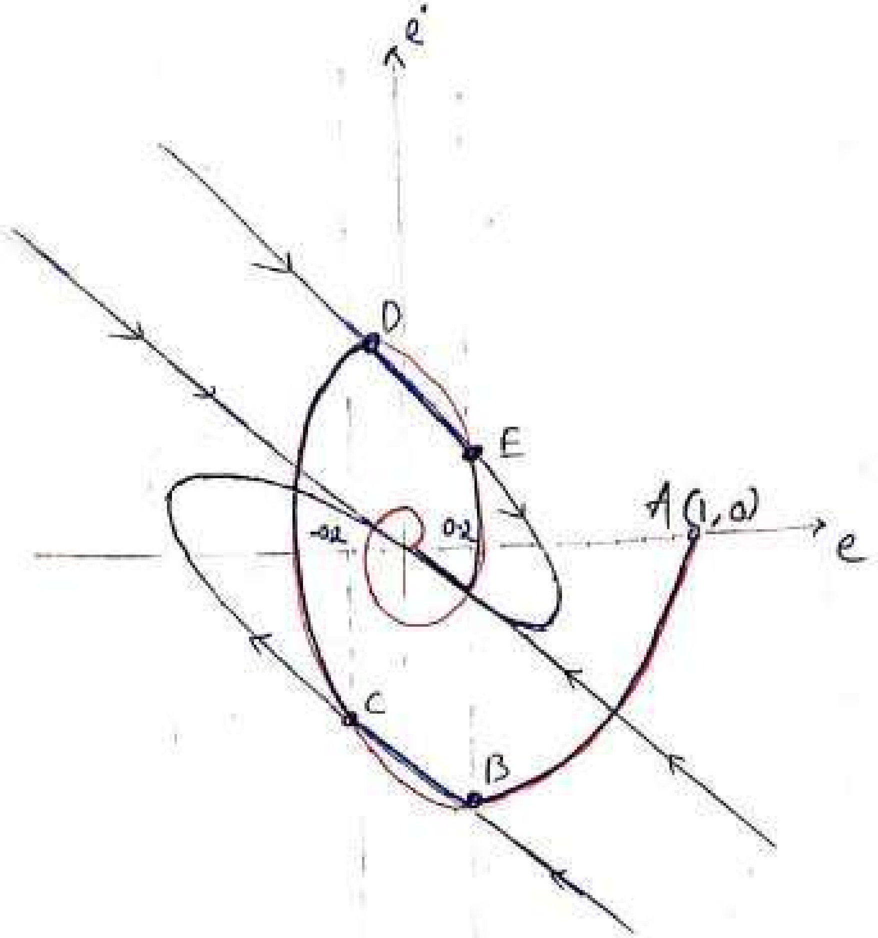
$$\lambda = \frac{-1 \pm \sqrt{1 - 4 \times 0.25}}{2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{0} = -\frac{1}{2} = -0.5$$

so, $\lambda_1 = -0.5$ and $\lambda_2 = -0.5$

so for $|e| < e_0$ the system is overdamped having double roots at -0.5.



(fig-2)



Analysis

Let the system be excited with unit step input.

$R = 1 \Rightarrow \dot{r} = 0$ and $\ddot{r} = 0$

As $e = r - c$

$e = 1 - 0 = 1$ [As system is initially at rest]

$\dot{e} = 0$

So the starting point of phase trajectory is $(1, 0)$ i.e. pt A

As $|e| > E_0$ i.e. $e > 0.2$

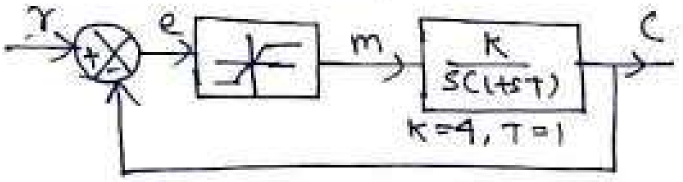
So, the system will take the phase trajectory of fig-1 and follow it until pt B is reached.

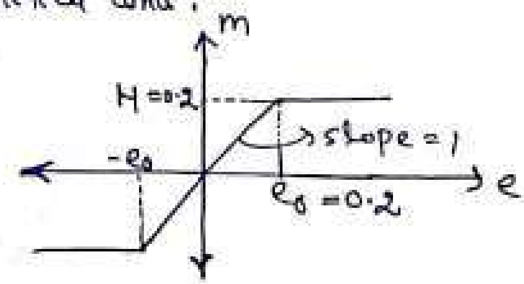
At point B, $e = 0.2$, so the system dynamics will be governed by phase trajectory shown in fig-2. So the system will follow the path BC.

After the point C, the |error| will be greater than 0.2. So again the phase trajectory will move along the trajectory shown in fig-1. Hence it follows the path CD.

At point D, $|e| = 0.2$. So phase trajectory move along the trajectory shown in fig-2 and finally reaches the equilibrium point $(0,0)$.

Q) Consider the position control system with a saturating amplifier, approximated by straight line segment as shown below. Draw the phase portrait for the system for step input $r(t) = R$. Assume zero initial condⁿ.





<u>Step-1</u> The non-linear block can be modelled as

$$m = \begin{cases} e, & |e| < 0.2 \\ M, & e > 0.2 \\ -M, & e < -0.2 \end{cases}$$

<u>Step-2</u> $$\frac{C}{m} = \frac{K}{S(1+ST)}$$

$$CS^2 + CST = Km$$

$$\ddot{c} + T\dot{c} = Km \quad \text{— (1)}$$

<u>Step-3</u> $e = r - c \Rightarrow c = r - e$

$$\dot{c} = \dot{r} - \dot{e}$$

$$\ddot{c} = \ddot{r} - \ddot{e}$$

Substituting the values of $\dot{c}$ & $\ddot{c}$ in eqⁿ-(1) we get

$$\ddot{r} - \ddot{e} + T(\dot{r} - \dot{e}) = Km$$

$$T\ddot{e} + \dot{e} + Km = T\ddot{r} + \dot{r} \quad \text{— (2)}$$

<u>Step-4</u> given that $r(t) = R$

so $\dot{r} = 0$ and $\ddot{r} = 0$ for $t > 0$

Hence eqⁿ-(2) is modified to

$$T\ddot{e} + \dot{e} + Km = 0 \quad \text{— (3)}$$

case-1

for $|e| < 0.2$ i.e. $-0.2 < e < 0.2$

$m = e$

so eq^n - (3) becomes

$$T\ddot{e} + \dot{e} + Ke = 0$$

Given that $T = 1$ and $K = 4$ so,

$$\ddot{e} + \dot{e} + 4e = 0 \quad -(4)$$

Assuming $x_1 = e$

$x_2 = \dot{e} = \dot{x}_1$

from eq^n-(4) $\dot{x}_2 + x_2 + 4x_1 = 0 \Rightarrow \dot{x}_2 = -4x_1 - x_2$

$\dot{x}_1 = 0 \cdot x_1 + 1 \cdot x_2$

$\dot{x}_2 = -4x_1 - 1 \cdot x_2$

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix}$$

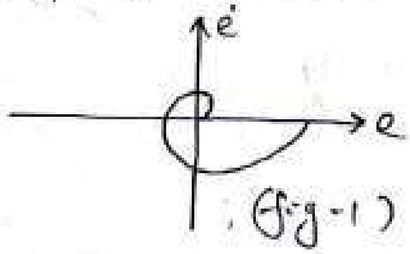
The eigen values of the system will be

$$|\lambda I - A| = 0$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \right| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ 4 & \lambda+1 \end{vmatrix} = 0$$

$$\lambda^2 + \lambda + 4 = 0 \Rightarrow \lambda = \frac{-1 \pm \sqrt{1-16}}{2} = -0.5 \pm \frac{1}{2}\sqrt{15}\, i$$

Since the $|A| \neq 0$ and eigen values are complex with (-)ve real part, origin will be the equilibrium point and the singular point will be a stable focus.



(fig-1)

case-2

for $e > 0.2$, $m = M = 0.2$

so eq^n-(3) becomes

$$T\ddot{e} + \dot{e} + KM = 0$$

given that $T = 1$, $K = 4$ & $M = 0.2$

$$\ddot{e} + \dot{e} + 0.8 = 0 \quad -(5)$$

Let us define the slope of the phase trajectory be α

i.e. $\alpha = \frac{\ddot{e}}{\dot{e}} \Rightarrow \ddot{e} = \alpha \dot{e}$

Substituting the value of $\ddot{e}$ in eqn -(5) we get

$$\alpha \dot{e} + \dot{e} + 0.8 = 0$$

$$\Rightarrow (1+\alpha)\dot{e} = -0.8$$

$$\Rightarrow \dot{e} = \frac{-0.8}{1+\alpha}$$

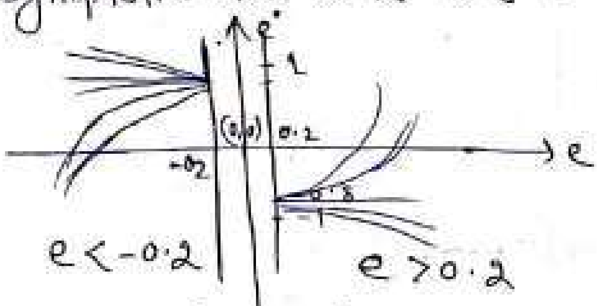
For constant value of e, $\ddot{e} = 0$ so $\alpha = 0$

Hence the phase trajectory in $e-\dot{e}$ plane will be asymptotic to the line -0.8. [As $\alpha = 0$, $\dot{e} = \frac{-0.8}{1+0} = -0.8$]

Case - 3

For $e < -0.2$, $m = -M = -0.2$

Proceeding as ~~case~~ 2, we will get the phase trajectory will be asymptotic the line 0.8.



(fig-2)

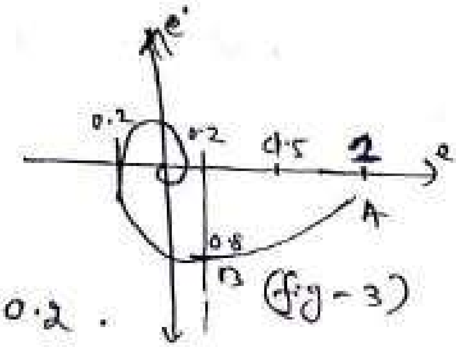
Analysis

Assuming $R = 1$

$e(0) = 1 - c(0) = 1 - 0 = 1$
$\dot{e}(0) = 0$ & $\ddot{e}(0) = 0$

[As system is assumed to be initially rest]

So the starting point is $(1, 0)$. In this region the characteristic is governed by eqn -(5) as $e > 0.2$. So the trajectory follow the path AB. ~~[illegible]~~ At B, the trajectory will be governed by eqn - 4 and moves along the trajectory as shown in figure-1 and finally reaches the singular point



(fig-3)

Liapunov's 2nd method / Direct method

Let us consider a system described by

$$\dot{x} = f(x,t)$$

$$f(0,t) = 0 \text{ for all } t$$

If there exists a scalar function $V(x,t)$ having continuous first partial derivatives and satisfying the conditions

$V(x,t)$ is positive definite

$\dot{V}(x,t)$ is negative definite.

then the equilibrium state at the origin is uniformly asymptotically stable.

Sign definiteness

(i) Positive definite: A scalar function $V(x)$ is said to be positive definite in a region Ω if $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$ for $x = 0$.

(ii) Positive semi-definite: A scalar function is said to be positive semidefinite if it is positive for all states in the region Ω except at the origin and at certain other states where it is zero.

(iii) Negative definite: A scalar function $V(x)$ is negative definite if $-V(x)$ is positive definite.

(iv) Negative semi-definite: A scalar function is said to be (−)ve semi-definite if $-V(x)$ is (+)ve semi definite.

(v) Indefiniteness: A scalar function $V(x)$ is said to be indefinite if in the region Ω it assumes both positive and negative values.

Sylvester's criteria

The necessary and sufficient condition that a quadratic function $V(x) = x^T P x$ is positive definite if all the principal minor of real, symmetric matrix P be positive.

Q) verify the sign definiteness of a quadratic function

$$Q = 10x_1^2 + 4x_2^2 + x_3^2 + 2x_2x_1 - 2x_2x_3 - 4x_1x_3$$

Step-1 Here Q is a function of $X = x_1, x_2$ & x_3.

So Q can be expressed as

$$Q = x^T P x$$

$$Q = [x_1 \; x_2 \; x_3] \begin{bmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$P = \begin{bmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

$$[10] > 0$$

$$\begin{bmatrix} 10 & 1 \\ 1 & 4 \end{bmatrix} = 39 > 0$$

$$\begin{bmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix} = 17 > 0$$

Hence the given function is positive definite.

NOTE:

→ $V(x) = x^T P x$ would have been ~~nega~~ positive semi definite if P was singular and all the principal minors of P were non-negative.

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Asymptotic stability analysis

consider a linear time-invariant system

$$\dot{X} = AX \;;\; |A| \neq 0$$

Then the only equilibrium state is the origin i.e. $X=0$

Let us choose a possible liapunov function as

$$V(x) = x^T P x$$

where P is positive definite real symmetric matrix.

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$$

$$= (Ax)^T P x + x^T P \dot{x}$$

$$= x^T (A^T P + PA) x$$

Let us assume $Q = -(A^T P + PA)$

If Q is positive definite then $\dot{V}(x)$ will be negative.

Hence the system will be stable.

Q) consider the dynamics of the system represented by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Formulate Liapunov function to test the asymptotic stability of the system.

Solⁿ: Let $V(x) = x^T P X$.

where $P = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ (P is choosen arbitrarily but it should be real, symmetric & positive definite)

$$Q = -(A^T P + PA)$$

$$= -\left[\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{bmatrix} + \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since Q is (+)ve definite, $\dot{V}(x)$ will be negative. So ~~that~~ $x^T P x$ is the required Liapunov function for the given system.

Q) A system is described as

$$\dot{x}_1 = -x_1 + x_2 + x_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = -x_1 - x_2 + x_2(x_1^2 + x_2^2)$$

Determine the asymptotic stability using Liapunov's 2nd method while the function

$$V(x) = x_1^2 + x_2^2$$

<u>solⁿ:</u> $V(x) = x^T P x - x_1^2 + x_2^2$

$$V(x) = [x_1 \ x_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Q = -[A^T P + PA] =$$

So $V(x)$ is the required Liapunov function.

$$V(x) = x_1^2 + x_2^2$$

$$\dot{V}(x) = \frac{dV}{dx_1} \times \frac{dx_1}{dt} + \frac{dV}{dx_2} \times \frac{dx_2}{dt}$$

$$= \dot{x}_1 \times 2x_1 + \dot{x}_2 * 2x_2$$

$$= 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2$$

Substituting the values of $\dot{x}_1$ & $\dot{x}_2$ in the previous eqⁿ

$$\dot{V}(x) = 2x_1[-x_1 + x_2 + x_1(x_1^2 + x_2^2)] + 2x_2[-x_1 - x_2 + x_2(x_1^2 + x_2^2)]$$

$$= 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1)$$

$\dot{V}(x)$ will be negative if $(x_1^2 + x_2^2) < 1$.

Q) Determine the stability of the system dynamics

$$\dot{x}_1 = -x_1 - 2x_2 + 2$$

$$\dot{x}_2 = x_1 - 4x_2 - 1.$$

Q) Determine the range of K by applying the Liapunov's 2nd method for the given system dynamics.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 + x_3$$
$$\dot{x}_3 = -Kx_1 - 4x_3$$
$$V(x) = 5Kx_1^2 + 2Kx_1x_2 + 20x_2^2 + 8x_2x_3 + x_3^2$$

Soln:

$$V(x) = 5Kx_1^2 + 2Kx_1x_2 + 20x_2^2 + 8x_2x_3 + x_3^2$$
$$\dot{V}(x) = 10Kx_1\dot{x}_1 + 2Kx_1\dot{x}_2 + 2K\dot{x}_1x_2 + 40x_2\dot{x}_2 + 8x_2\dot{x}_3 + 8\dot{x}_2x_3 + 2x_3\dot{x}_3$$

Putting the values of $\dot{x}_1$, $\dot{x}_2$ & $\dot{x}_3$ in the previous eqn

$$\dot{V}(x) = -(40 - 2K)x_2^2$$

So $\dot{V}(x)$ to be negative $40 - 2K > 0$

$$\Rightarrow \boxed{K < 20}$$

Again $V(x) = x^T P x = 5Kx_1^2 + 2Kx_1x_2 + 20x_2^2 + 8x_2x_3 + x_3^2$

$$P = \begin{bmatrix} 5K & K & 0 \\ K & 20 & 4 \\ 0 & 4 & 1 \end{bmatrix}$$

Applying sylvester's criteria,

$$5K > 0 \Rightarrow K > 0$$

$$\begin{bmatrix} 5K & K \\ K & 20 \end{bmatrix} = 100K - K^2 > 0$$
$$(100 - K) > 0$$
$$\Rightarrow \boxed{K < 100}$$

$$\begin{bmatrix} 5K & K & 0 \\ K & 20 & 4 \\ 0 & 4 & 1 \end{bmatrix}$$

Describing Function

The describing function of a non-linear element is defined to be the complex ratio of the fundamental harmonic component of the output to the input.

$$\boxed{N = \frac{Y_1}{X} \angle \phi_1}$$

where, $X \rightarrow$ Amplitude of i/p sinusoid

$Y_1 \rightarrow$ Amplitude of fundamental harmonic component of the output.

$\phi_1 \rightarrow$ phase shift of the fundamental harmonic component of the o/p wrt the i/p.

If the input to the system is $x = X \sin\omega t$

Then the fundamental component of the o/p will be

$$y_1(t) = a_0 + a_1 \cos\omega t + b_1 \sin\omega t$$

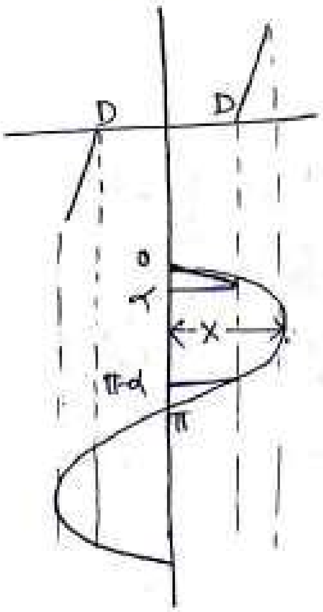
where, $a_0 = \frac{1}{\pi} \int_0^{2\pi} y_1(t)\, d\omega t$

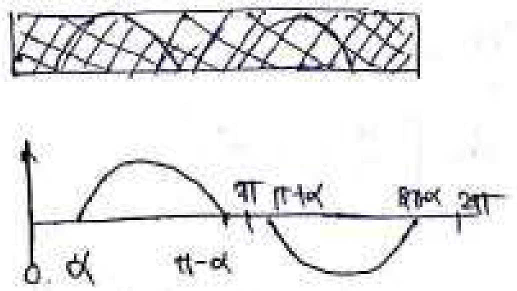
$$a_1 = \frac{1}{\pi} \int_0^{2\pi} y(t) \cos\omega t \; d(\omega t)$$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} y(t) \sin\omega t \; d(\omega t)$$

$$Y_1 = \sqrt{a_1^2 + b_1^2} \quad \text{and} \quad \phi_1 = \tan^{-1}\left(\frac{a_1}{b_1}\right)$$

Describing Function of Dead zone Non linearity





Since the o/p wave exhibit odd symmetry as well as half wave symmetry,

$$a_0 = 0$$

$$a_1 = 0$$

$$b_1 = \frac{4}{\pi}\int_0^{\pi/2} y(t)\sin\omega t \, d(\omega t)$$

$$y(t) = \begin{cases} 0 & 0 < \omega t < \alpha \\ K(X\sin\omega t - D) & \alpha < \omega t < \pi - \alpha \\ 0 & \pi - \alpha < \omega t < \pi \end{cases}$$

$$b_1 = \frac{4}{\pi}\int_\alpha^{\pi/2} (X\sin\omega t - D)\sin\omega t \, d(\omega t)$$

$$b_1 = \frac{2KX}{\pi}\left[\frac{\pi}{2} - \alpha - \frac{D}{X}\cos\alpha\right] \quad \text{---(1)}$$

from the o/p figure,

$$X\sin\alpha = D \Rightarrow \sin\alpha = \frac{D}{X} \Rightarrow \alpha = \sin^{-1}\left(\frac{D}{X}\right)$$

$$\cos\alpha = \sqrt{1 - \left(\frac{D}{X}\right)^2}$$

substituting the values of α & $\cos\alpha$ in eqn-(1)

$$b_1 = \frac{2KX}{\pi}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{D}{X}\right) - \frac{D}{X}\sqrt{1-\left(\frac{D}{X}\right)^2}\right]$$

$$Y_1 = \sqrt{a_1^2 + b_1^2} = b_1$$

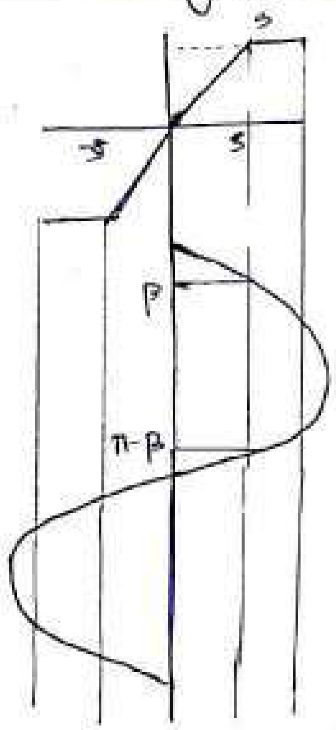
$$\phi_1 = \tan^{-1}\left(\frac{a_1}{b_1}\right) = \tan^{-1}\left(\frac{0}{b_1}\right) = 0$$

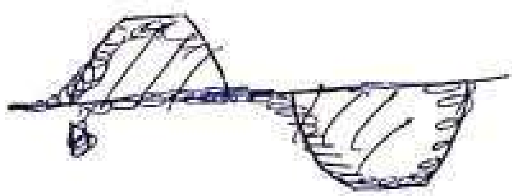
$$N = \frac{Y_1}{X}\angle\phi_1$$

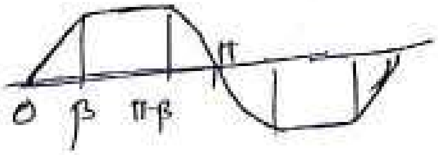
$$N = \frac{2K}{\pi}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{D}{X}\right) - \frac{D}{X}\sqrt{1-\left(\frac{D}{X}\right)^2}\right]\angle 0^\circ$$

$$\boxed{N = \left[K - \frac{2K}{\pi}\left[\sin^{-1}\left(\frac{D}{X}\right) + \frac{D}{X}\sqrt{1-\left(\frac{D}{X}\right)^2}\right]\right]\angle 0^\circ}$$

Describing function for Saturation







$$x = X \sin \omega t$$

$$y = \begin{cases} KX \sin \omega t & 0 < \omega t < \beta \\ KS &, \beta < \omega t < \pi - \beta \\ KX \sin \omega t &, \pi - \beta < \omega t < \pi \end{cases}$$

$$X \sin\beta = S \quad \Rightarrow \sin\beta = \frac{S}{X}$$

$$\beta = \sin^{-1}\left(\frac{S}{X}\right)$$

$$\cos\beta = \sqrt{1 - \left(\frac{S}{X}\right)^2}$$

Since the o/p waveform exhibit odd symmetry as well as half wave symmetry,

$$a_0 = 0$$

$$a_1 = 0$$

$$b_1 = \frac{4}{\pi} \int_{\beta}^{\pi/2} y(t) \sin \omega t \, d(\omega t)$$

$$= \frac{4}{\pi} \int_0^{\beta} KX \sin \omega t \sin \omega t \, d(\omega t) + \frac{4}{\pi} \int_{\beta}^{\pi/2} KS \sin \omega t \, d(\omega t)$$

$$b_1 = \frac{2KX}{\pi} \left[\beta + 2\frac{S}{X} \cos\beta - \sin\beta \cos\beta \right] \quad \text{---(1)}$$

Substituting the value of β, $\sin\beta$ & $\cos\beta$ in eqn-(1) we get

$$b_1 = \frac{2KX}{\pi} \left[\sin^{-1}\left(\frac{S}{X}\right) + 2 \cdot \frac{S}{X} \times \sqrt{1 - \left(\frac{S}{X}\right)^2} - \frac{S}{X} \sqrt{1 - \left(\frac{S}{X}\right)^2} \right]$$

$$= \frac{2KX}{\pi} \left[\sin^{-1}\left(\frac{S}{X}\right) + \frac{S}{X} \sqrt{1 - \left(\frac{S}{X}\right)^2} \right]$$

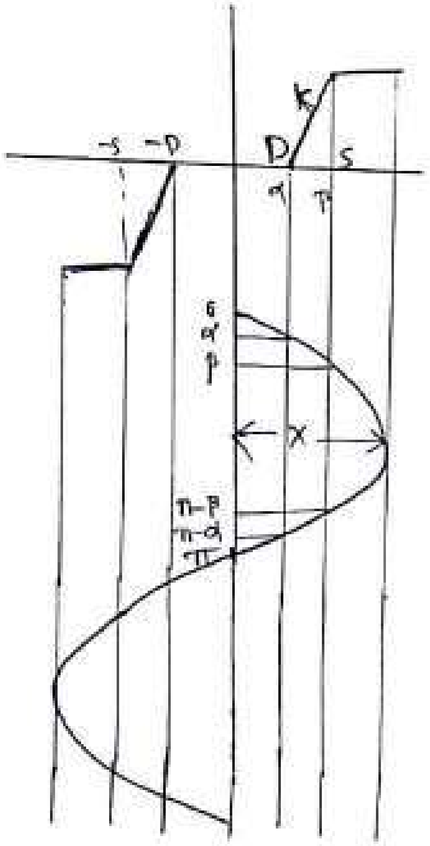
$$Y_1 = \sqrt{a_1^2 + b_1^2} = b_1$$

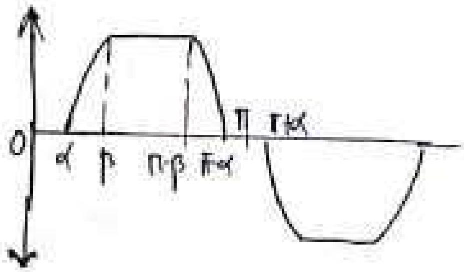
$$\phi_1 = \tan^{-1}\left(\frac{a_1}{b_1}\right) = \tan^{-1}\left(\frac{0}{b_1}\right) = 0$$

$$N = \frac{Y_1}{X} \angle \phi_1$$

$$\boxed{N = \frac{2K}{\pi}\left[\sin^{-1}\left(\frac{S}{X}\right) + \frac{S}{X}\sqrt{1-\left(\frac{S}{X}\right)^2}\right] \angle 0^\circ}$$

Describing Function of Deadzone & Saturation Non-linearity





$$D = X\sin\alpha \Rightarrow \alpha = \sin^{-1}\left(\frac{D}{X}\right)$$

$$S = X\sin\beta \Rightarrow \beta = \sin^{-1}\left(\frac{S}{X}\right)$$

$$y(t) = \begin{cases} 0 & 0 < \omega t < \alpha \\ K(X\sin\omega t - D) & \alpha < \omega t < \beta \\ K(S-D) & \beta < \omega t < \pi-\beta \\ K(X\sin\omega t - D) & \pi-\beta < \omega t < \pi-\alpha \\ 0 & \pi-\alpha < \omega t < \pi \end{cases}$$

Since the o/p wave exhibits odd symmetry and half wave symmetry,

$$a_0 = 0$$

$$a_1 = 0$$

$$b_1 = \frac{4}{\pi}\int_0^{\pi/2} y(t)\sin\omega t \, d(\omega t)$$

$$b_1 = \frac{4}{\pi}\left[\int_0^{\alpha} 0\cdot\sin\omega t \cdot d(\omega t) + \int_{\alpha}^{\beta} K(X\sin\omega t - D)\cdot\sin\omega t \, d(\omega t) + \int_{\beta}^{\pi/2}(KS-D)\sin\omega t \cdot d(\omega t)\right]$$

$$b_1 = \frac{KX}{\pi}\left[2(\beta-\alpha) + (\sin 2\beta - \sin 2\alpha)\right]$$

$$b_1 = \frac{KX}{\pi}\left[2(\beta-\alpha) + 2\sin\beta\cos\beta - 2\sin\alpha\cdot\cos\alpha\right]$$

$$b_1 = \frac{2KX}{\pi}\left[(\beta-\alpha) + \sin\beta\cdot\cos\beta - \sin\alpha\cdot\cos\alpha\right]$$

Substituting $\alpha = \sin^{-1}\left(\frac{D}{X}\right)$

$\beta = \sin^{-1}\left(\frac{S}{X}\right)$

$\sin\beta = \frac{S}{X}$, $\cos\beta = \sqrt{1-\left(\frac{S}{X}\right)^2}$

$\sin\alpha = \frac{D}{X}$, $\cos\alpha = \sqrt{1-\left(\frac{D}{X}\right)^2}$

$$b_1 = \frac{2KX}{\pi}\left[\sin^{-1}\left(\frac{S}{X}\right) - \sin^{-1}\left(\frac{D}{X}\right) + \frac{S}{X}\sqrt{1-\left(\frac{S}{X}\right)^2} - \frac{D}{X}\sqrt{1-\left(\frac{D}{X}\right)^2}\right]$$

$$Y_1 = \sqrt{a_1^2 + b_1^2} = b_1$$

$$\phi = \tan^{-1}\left(\frac{a_1}{b_1}\right) = \tan^{-1}\left(\frac{0}{b_1}\right) = 0^\circ$$

$$N = \frac{Y_1}{X}\angle 0^\circ$$

$$\boxed{N = \frac{2K}{\pi}\left[\sin^{-1}\left(\frac{S}{X}\right) - \sin^{-1}\left(\frac{D}{X}\right) + \frac{S}{X}\sqrt{1-\left(\frac{S}{X}\right)^2} - \frac{D}{X}\sqrt{1-\left(\frac{D}{X}\right)^2}\right]\angle 0^\circ}$$

case-1

For only saturation non-linearity, $D=0$ & $\alpha=0$

so, $N = \frac{2K}{\pi}\left[\sin^{-1}\left(\frac{S}{X}\right) - \sin^{-1}(0) + \frac{S}{X}\sqrt{1-\left(\frac{S}{X}\right)^2} - 0\right]\angle 0^\circ$

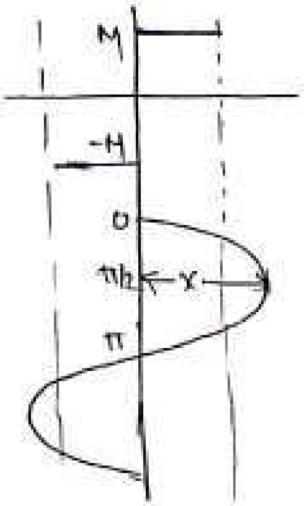
$$\boxed{N = \frac{2K}{\pi}\left[\sin^{-1}\left(\frac{S}{X}\right) + \frac{S}{X}\sqrt{1-\left(\frac{S}{X}\right)^2}\right]\angle 0^\circ}$$

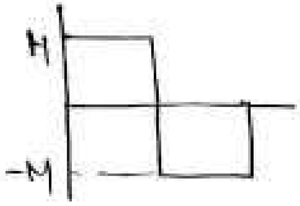
case-2

For only dead-zone non-linearity, $S \to \infty$ & $\beta = \frac{\pi}{2}$

$$\boxed{N = \frac{2K}{\pi}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{D}{X}\right) - \frac{D}{X}\sqrt{1-\left(\frac{D}{X}\right)^2}\right]\angle 0^\circ}$$

Describing function for Ideal Relay (ON/OFF) Non-linearity





$$y(t) = \begin{cases} M & 0 \le \omega t \le \pi \\ -M & \pi \le \omega t \le 2\pi \end{cases}$$

Since the o/p wave shows odd symmetry and half wave symmetry,

$$a_0 = 0$$

$$a_1 = 0$$

$$b_1 = \frac{4}{\pi} \int_0^{\pi/2} y(t) \sin\omega t \cdot d(\omega t)$$

$$b_1 = \frac{4}{\pi} \int_0^{\pi/2} M \sin\omega t \, d(\omega t) = \frac{4M}{\pi}$$

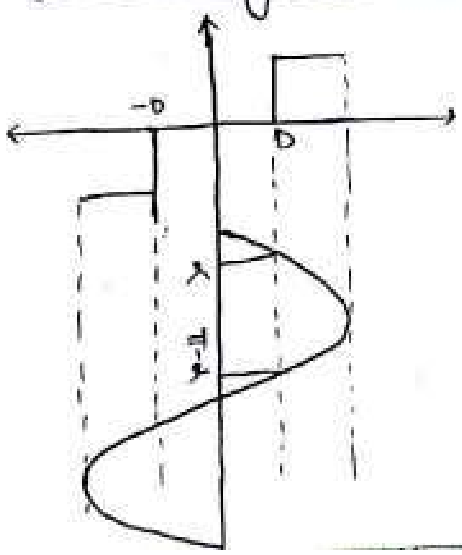
$$Y_1 = \sqrt{a_1^2 + b_1^2} = b_1$$

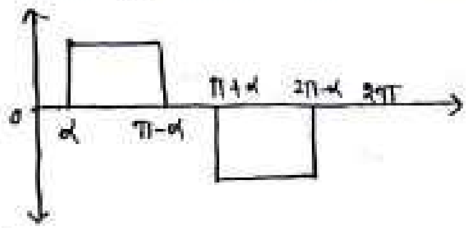
$$\phi = \tan^{-1}\left(\frac{a_1}{b_1}\right) = 0^\circ$$

$$N = \frac{Y_1}{X} \angle\phi = \frac{4M}{\pi} \times \frac{1}{X} \angle 0^\circ$$

$$\boxed{N = \frac{4M}{\pi X} \angle 0^\circ}$$

Describing Function for Relay with dead zone



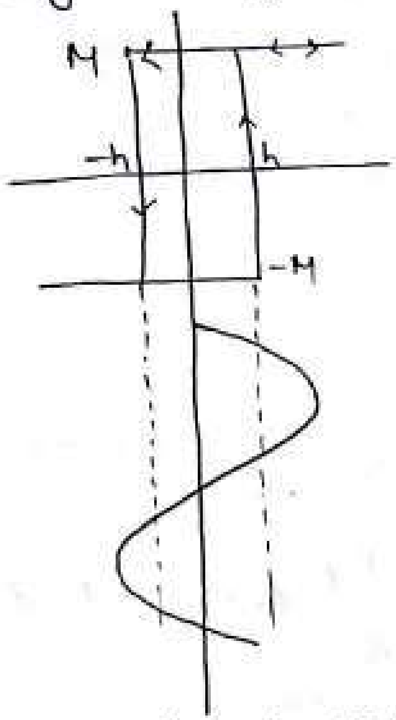


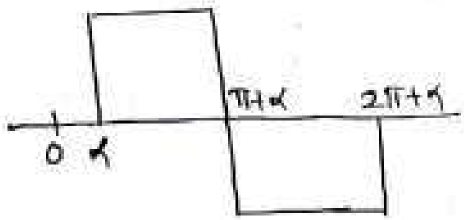
$$y(t) = \begin{cases} 0 & 0 < \alpha < \omega t \\ M & \alpha < \omega t < \pi - \alpha \\ 0 & \pi - \alpha < \omega t < \pi \end{cases}$$

Since the o/p waveform shows odd symmetry and half wave symmetry,

$$a_0 = 0$$
$$a_1 = 0$$
$$b_1 = \frac{4}{\pi}\int_0^{\pi/2} y(t)\sin\omega t\, d(\omega t)$$
$$b_1 = \frac{4}{\pi}\left[\int_\alpha^{\pi/2} M \sin\omega t\, d(\omega t)\right]$$
$$= \frac{4M}{\pi}\cos\alpha = \frac{4M}{\pi}\left[1-\left(\frac{D}{X}\right)^2\right]^{\frac{1}{2}}$$
$$Y_1 = \sqrt{a_1^2 + b_1^2} = b_1$$
$$\phi = \tan^{-1}\left(\frac{a_1}{b_1}\right) = 0$$
$$\boxed{N = \frac{Y_1}{X}\angle\phi_1 = \frac{4M}{\pi X}\left[\sqrt{1-\left(\frac{D}{X}\right)^2}\right]\angle 0^\circ}$$

Relay with Hysteresis





$$y = \begin{cases} M & \alpha < \omega t < \pi+\alpha \\ -M & \pi+\alpha < \omega t < 2\pi+\alpha \end{cases}$$

$$h = X\sin\alpha \Rightarrow \boxed{\alpha = \sin^{-1}\left(\frac{h}{X}\right)}$$

Since the o/p wave exhibits odd symmetry & half wave symmetry,

$$a_0 = 0$$
$$a_1 = 0$$
$$b_1 = \frac{2}{\pi}\int_\alpha^{\pi+\alpha} y(t)\sin\omega t\, d(\omega t)$$

$$b_1 = \frac{2}{\pi}\int_{\alpha}^{\pi+\alpha} M \sin\omega t \, d(\omega t) = \frac{4M}{\pi}$$

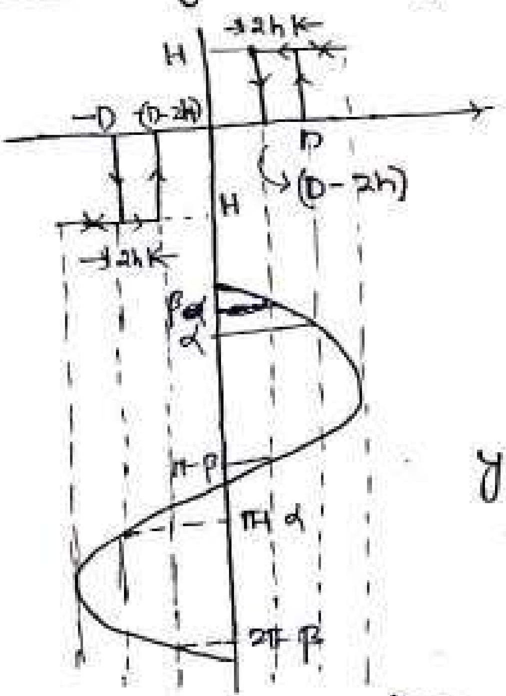
Due to hysteresis, the o/p wave lags the i/p by

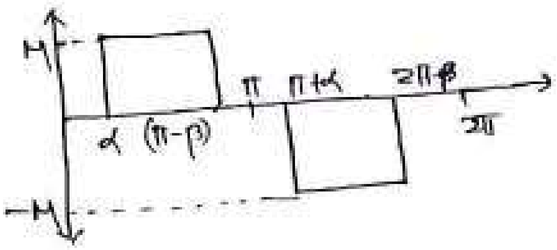
$$\alpha = \sin^{-1}\left(\frac{h}{X}\right)$$

Hence the describing function of relay with hysteresis will be

$$\boxed{N = \frac{4M}{\pi X}\angle -\sin^{-1}\left(\frac{h}{X}\right)}$$

Describing for Relay with Dead zone & Hysteresis





$$y(t) = \begin{cases} 0 ; & 0 \le \omega t \le \alpha \\ M ; & \alpha \le \omega t \le (\pi-\beta) \\ 0 ; & (\pi-\beta) \le \omega t \le (\pi+\alpha) \\ -M ; & (\pi+\alpha) \le \omega t \le (2\pi-\beta) \\ 0 ; & (2\pi-\beta) \le \omega t \le 2\pi \end{cases}$$

$$\alpha = \sin^{-1}\left(\frac{D}{X}\right)$$

$$\beta = \sin^{-1}\left(\frac{D-2h}{X}\right)$$

$a_0 = 0$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} y \cos\omega t \, d(\omega t) = \frac{2}{\pi}\int_{\alpha}^{\pi-\beta} M \cos\omega t \, d(\omega t)$$

$$a_1 = \frac{2M}{\pi}\left(\frac{-2h}{X}\right) = -\frac{4Mh}{\pi X}$$

$$\frac{a_1}{X} = \frac{2M}{\pi X}\left(\frac{-2h}{X}\right)$$

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$$b_1 = \frac{2}{\pi}\int_{\alpha}^{\pi-\beta} M \sin\omega t \, d(\omega t) = \frac{2M}{\pi}(\cos\alpha + \cos\beta)$$

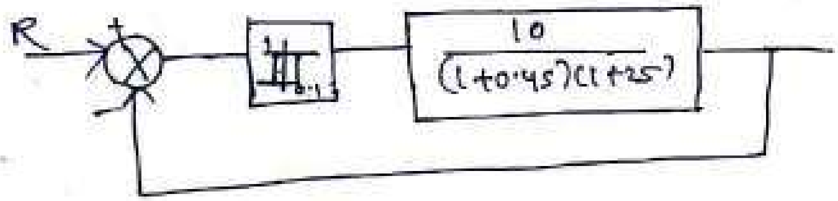
$$= \frac{2M}{\pi}\left[\sqrt{1-\left(\frac{D}{X}\right)^2} + \sqrt{1-\left(\frac{D-2h}{X}\right)^2}\right]$$

$$\frac{b_1}{X} = \frac{2M}{\pi X}\left[\sqrt{1-\left(\frac{D}{X}\right)^2} + \sqrt{1-\left(\frac{D-2h}{X}\right)^2}\right]$$

$$\boxed{N = \sqrt{\left(\frac{a_1}{X}\right)^2 + \left(\frac{b_1}{X}\right)^2} \angle \tan^{-1}\left(\frac{a_1}{b_1}\right)} \quad |x| > D$$

Q) For the following Non-linear system determine the amplitude and freq. of limit cycle. The describing function of non-linearity is given by,

$$N = \frac{4M}{\pi X} \angle -\sin^{-1}(h/X)$$



<u>Step-1</u> $N = \frac{4M}{\pi X} \angle -\sin^{-1}(h/X)$

given $h = 0.1$ and $M = 1$

$$N = \frac{4 \times 1}{\pi X} \angle -\sin^{-1}(0.1/X)$$

$$\boxed{N = \frac{4}{\pi X} \angle -\sin^{-1}(0.1/X)}$$

<u>Step-2</u>

$$1 + G(s) N = 0$$

$$\Rightarrow G(s) = -\frac{1}{N}$$

<u>Step-3</u>

$$\frac{-1}{N} = \frac{\pi X}{4} \angle -180 + \sin^{-1}(0.1/X)$$

X		$-1/N$
0.1	$\rightarrow$	$0.0785 \angle -90$
0.2	$\rightarrow$	$0.157 \angle -150$
0.3	$\rightarrow$	$0.235 \angle -160.5$
0.5	$\rightarrow$	$0.393 \angle -168.5$
0.8	$\rightarrow$	$0.628 \angle -172.8$
9.0	$\rightarrow$	$0.785 \angle -174.3$

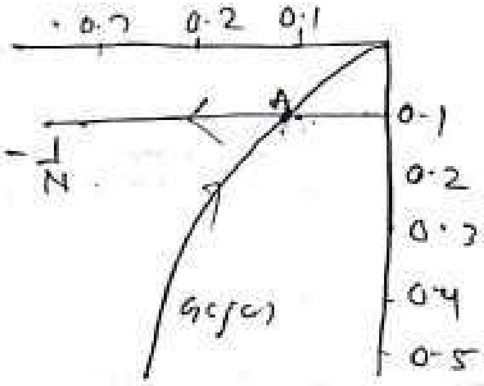
Step-4

$$G(s) = \frac{10}{(1+0.4s)(1+2s)}$$

$$G(j\omega) = \frac{10}{(1+j0.4\omega)(1+j2\omega)}$$

$$|G(j\omega)| = \frac{10}{\sqrt{1+0.16\omega^2}\sqrt{1+4\omega^2}}$$

$$\angle G(j\omega) = -\tan^{-1} 0.4\omega - \tan^{-1} 2\omega$$



ω		$G(j\omega)$
3	$\longrightarrow$	$1.052 \angle -130.7$
4	$\longrightarrow$	$0.057 \angle -140.8$
6	$\longrightarrow$	$0.319 \angle -152.6$
7	$\longrightarrow$	$0.239 \angle -156.6$
8	$\longrightarrow$	$0.186 \angle -159$
10	$\longrightarrow$	$0.121 \angle -163.1$
20	$\longrightarrow$	$0.031 \angle -171.4$

The curve of $-\frac{1}{N}$ and $G(j\omega)$ intersects at a point A. which corresponds to stable limit cycle.

$$-\frac{1}{N} = 0.21 \quad [\text{from graph}]$$

$$\frac{\pi X}{4} = 0.21 \Rightarrow X = 0.267$$

$$|G(j\omega)| = \frac{10}{\sqrt{1+0.16\omega^2}\sqrt{1+4\omega^2}} = 0.21 \Rightarrow \omega = 7.92 \text{ rad/sec.}$$

PHASE PLANE ANALYSIS :-

The phase plane method is a graphical method of solving the 2nd order non-linear differential eqn.

The co-ordinate plane corresponding to x and $\dot{x}$ is called phase plane.

The curve described in phase plane wrt time is called as phase trajectory.

The family of trajectory for different initial condn is k/as phase portrait. The phase portrait gives the information about the stability and limit cycles.

Construction of phase trajectory using method of isoclines :-

Isoclines are the lines in the phase-plane corresponding to slopes of the phase portrait. close spacing of isoclines ~~result~~ increases the accuracy of resulting trajectory.

A 2nd order system can be defined as

$$\ddot{E} + 2\delta\omega_n\dot{E} + \omega_n^2 E = 0$$

$$\frac{d\dot{E}}{dt} + 2\delta\omega_n\frac{dE}{dt} + \omega_n^2 E = 0$$

Diving $\dot{E}$ in the previous eqn

$$\frac{1}{\dot{E}}\frac{d\dot{E}}{dt} + 2\delta\omega_n + \omega_n^2\frac{E}{\dot{E}} = 0$$

Putting $\ddot{E} = N = \tan\phi$ = slope of the trajectory

$$\frac{1}{\dot{E}}N + 2\delta\omega_n + \omega_n^2\frac{E}{\dot{E}} = 0$$

$$N = -2\delta\omega_n - \omega_n^2\frac{E}{\dot{E}}$$

$$\frac{\dot{E}}{E} = \frac{-\omega_n^2}{N + 2\delta\omega_n}$$

Q) Draw the trajectory when $\delta = 0.5$ $\omega_n = 1$ rad/sec.

We know that, $\frac{\dot{E}}{E} = \frac{-\omega_n^2}{N + 2\delta\omega_n}$.

$$\frac{\dot{E}}{E} = \frac{-1}{N+1} = \tan\theta$$

for unit step i/p, $\theta_r = 1$, $\theta_c = 0$

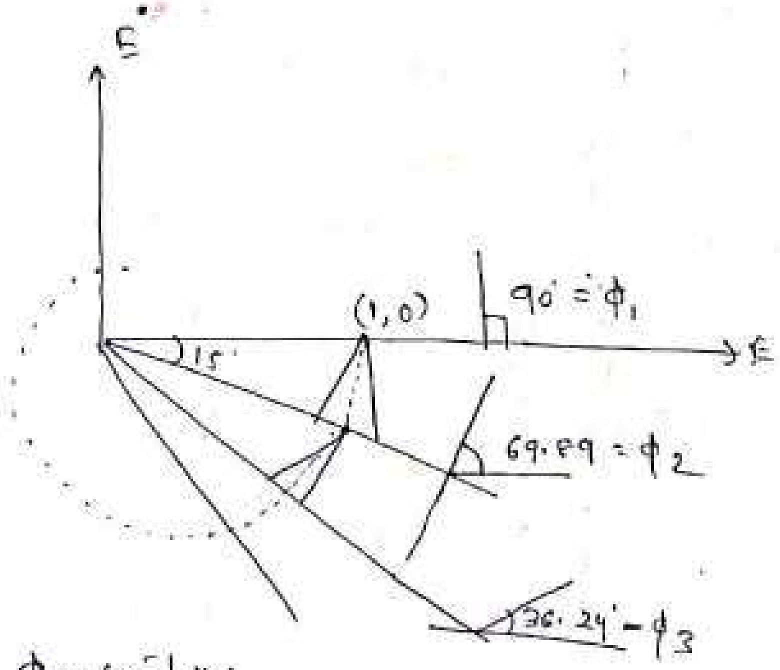
$$E = \theta_r - \theta_c = 1 - 0 = 1$$
$$\dot{E} = 0$$

Procedure for drawing phase trajectory.

→ Assume a suitable scale, 15 cm = 1 unit

→ Select θ at equidistance ; let $\theta = 15°$

→ Draw a line parallel to first slope marker from (1, 0) and from the same point draw a line parallel to 2nd slope marker, middle of these will give second point of the trajectory i.e. point on isocline 2.



$\theta = \tan^{-1}\frac{1}{N+1}$ $\phi = \tan^{-1} N$

52
43

θ	ϕ
0°	90
-15	69.59
-30	36.24