Module - II

Discrete Mathematics

4th Semester

Computer Science

RECURRENCE RELATION

Def^n A recurrence relation of the sequence $\{a_n\}$ is an equation that expresses 'a_n' in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, a_2 \ldots a_{n-1}$ $\forall$ integer n, with $n \geq n_0$ where n_0: non-ve integer.

- A recurrence relation are also called differential equation.

Ex- $a_{n+1} = a_n + a_{n-1}$ initial condition $a_1 = 2, a_2 = 3$.

$a_n = a_{n-1} + a_{n-2}$ $\quad a_0 = 0, a_1 = 1$

Solution of Recurrence Relation

A sequence is called a sol^n of a recurrence relation if its terms satisfy the recurrence relation.

Ex- Let $\{a_n\}$ be a sequence that satisfies that recurrence relation

$a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4 \ldots$

and suppose that $a_0 = 3$, and $a_1 = 5$ what are a_2 and a_3.

Sol^n- given that $a_n = a_{n-1} - a_{n-2}$

putting $n = 2, 3 \ldots$ Here $a_0 = 3$, $a_1 = 5$

So, $a_2 = a_1 - a_0 = 5 - 3 = 2$

$a_3 = a_2 - a_1 = 2 - 5 = -3$

Linear Recurrence Relation with constant coefficients (LRR with constant coefficient)

A recurrence relation is of the form

$$\boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)}$$

where $c_1, c_2 \ldots c_k$ are real numbers and $c_k \neq 0$. — (1)

is called linear R.R w. const. coefficients

The recurrence relation in eqn ① is known as k^{th} degree recurrence relation.

provided $c_k \neq 0$

Ex- 2nd degree R.R

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + F(n)$$

3rd degree R.R $\quad a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + F(n)$

Linear R.R. w. const coefficients

Linear Homogeneous R.R

If $F(n) = 0$ in eqn ①

i.e. $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$.

Linear Non Homogeneous R.R

$F(n) \neq 0$ in eqn ①.

$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + F(n)$

where $F(n)$ is polynomial or constant.

Ex- $F(n) = n^2, 2n, n^2+n+1, n^3$
$2^n, 3^n, 7^n, \ldots 5, 7, 1$
$2^n(n+1), n \cdot 3^n, n!$
$3^n+(n+1) \ldots$ etc.

Ex- $P_n = (1.11) P_{n-1}$ Linear H.R.R, degree = 1

$f_n = f_{n-1} + f_{n-2}$ Linear H.R.R, degree = 2

$a_n = a_{n-5}$ " " = 5

$a_n = n a_{n-1}$ not linear ($\because$ doesnot have constant coefficients

$a_n = a_{n-1} + a_{n-2}^2$ (not linear H.R.R)

$a_n = 2a_{n-1} + 1$ (not linear H.R.R) (Non Homogeneous R.R)

Linear Non Homogeneous R.R
$\begin{cases} a_n = a_{n-1} + 2^n \\ a_n = a_{n-1} + a_{n-2} + n^2 + n + 1 \\ a_n = a_{n-1} + a_{n-2} + a_{n-3} + n! \end{cases}$

$a_n = 3a_{n-1} + n3^n$

$a_n = 2a_{n-1} + 2^n(n+1)$

Solving Linear Homogeneous R.R with Constant Coefficients

Suppose, The 2nd degree Linear Homogeneous R.R is

$$\boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2}}$$

or, $a_n - c_1 a_{n-1} - c_2 a_{n-2}$

$\therefore$ The Characteristic equ^n is

$$r^2 - c_1 r - c_2 = 0$$

This sol^n of this eqn are called Characteristic root.

Case-I If the roots are real and unequal.

say, $r_1 \neq r_2$

$\therefore$ The sol^n of Recurrence relation (general sol^n) is

$$\boxed{a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n} \quad \text{For } n = 0, 1, 2, \ldots$$

where α_1, α_2 are Constants.

Similarly suppose degree '3'.

and eqn's having 3 roots. $r_1 \neq r_2 \neq r_3$

Then sol^n is $a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n + \alpha_3 \cdot r_3^n$

Suppose 'p' roots,

Then sol^n is $a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n + \cdots + \alpha_p r_p^n$

Case-II If the roots are real and equal.

Say, $r_1 = r_2 = r$

$\therefore$ The sol^n of R.R is $\boxed{a_n = (\alpha_1 + n\alpha_2) r^n}$

For $n = 0, 1, 2 \ldots$

and α_1 α_2 are constants.

Similarly, suppose degree 3 linear Homogeneous R.R.

and the roots are $r_1 = r_2 = r_3 = r$.

Then the Soln is $\boxed{a_n = (\alpha_1 + n\alpha_2 + n^2\alpha_3) r^n}$

For $n = 0, 1, 2 \ldots$

where $\alpha_1\ \alpha_2\ \alpha_3$ are constants.

Case III If the roots are complex number.

Say, $r = \alpha + i\beta$

Then the Soln is

$$\boxed{a_n = (\alpha_1 \cos n\theta + \alpha_2 \sin n\theta) R^n}$$

where $R = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$.

EX-1 Solve the Recurrence Relation

$a_n = a_{n-1} + 2a_{n-2}$

with $a_0 = 2$, and $a_1 = 7$.

Soln Given Recurrence Relation

$a_n = a_{n-1} + 2a_{n-2}$

This is 2nd degree R.R and $a_0 = 2$, $a_1 = 7$.

$\therefore$ The Characteristic eqn is

$$r^2 - r - 2 = 0$$

$$\Rightarrow r^2 - 2r + r - 2 = 0$$

$$\Rightarrow r(r-2) + 1(r-2) = 0 \Rightarrow (r-2)(r+1) = 0$$

$$\Rightarrow r = 2, -1$$

$\therefore$ The Soln of R.R is

$$\boxed{a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot (-1)^n}$$

where α_1 & α_2 are constants

but initial condition

$a_0 = 2$, For $n=0$, $a_0 = \alpha_1 + \alpha_2$

$2 = \alpha_1 + \alpha_2$ —— (2)

$n=1, \quad a_1 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$

$7 = 2\alpha_1 - \alpha_2$ —— (3)

solving eqn (2) & (3) we get

$$\begin{aligned} \alpha_1 + \alpha_2 &= 2 \\ 2\alpha_1 - \alpha_2 &= 7 \\ \hline 3\alpha_1 &= 9 \end{aligned}$$

$$\alpha_1 = 3$$

$\alpha_1 + \alpha_2 = 2$

$\Rightarrow \alpha_2 = 2 - 3 = -1 \Rightarrow \alpha_2 = -1$

∴ Hence the Soln of Recurrence relation is

$$a_n = 3 \cdot 2^n + (-1) \cdot (-1)^n$$

Ex-2 Solve $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

with $a_0 = 1$

$a_1 = -2, \ a_2 = -1$

Soln given R.R is

$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

$\Rightarrow a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$

∴ The characteristic eqn is

$$r^3 + 3r^2 + 3r + 1 = 0$$

$$\Rightarrow (r+1)^3 = 0$$

$$\Rightarrow r = -1, -1, -1$$

∴ The Soln of R.R is

$$a_n = (\alpha_1 + n\alpha_2 + n^2\alpha_3) \cdot r^n$$

$$\boxed{a_n = (\alpha_1 + n\alpha_2 + n^2\alpha_3) \cdot (-1)^n} \text{ —— (1)}$$

where $\alpha_1, \alpha_2, \alpha_3$ are constants.

but initial conditions

$a_0 = 1, \ a_1 = -2, \ \& \ a_2 = -1.$

For $n=0$, in eqn (1)

$a_0 = (\alpha_1 + 0 + 0) \cdot 1$

$\Rightarrow \boxed{1 = \alpha_1}$

For $n=1$, $a_1 = (\alpha_1+\alpha_2+\alpha_3)(-1)^1$

$$\Rightarrow -2 = -(\alpha_1+\alpha_2+\alpha_3)$$

$$\Rightarrow \boxed{\alpha_1+\alpha_2+\alpha_3 = 2} \quad \text{--- (2)}$$

For $n=2$, $a_2 = (\alpha_1 + 2\alpha_2 + 4\alpha_3)(-1)^2$

$$\Rightarrow \boxed{-1 = \alpha_1 + 2\alpha_2 + 4\alpha_3} \quad \text{--- 3}$$

Solving eqn (2) & (3)

$$2(\alpha_1+\alpha_2+\alpha_3) = 2$$

$$\alpha_1 + 2\alpha_2 + 4\alpha_3 = -1$$

$$(-) \quad (-) \quad (-) \quad (+)$$

$$\alpha_1 - 2\alpha_3 = 5$$

$$\Rightarrow -2\alpha_3 = 5-1 = 4$$

$$\Rightarrow \alpha_3 = -2$$

$$\alpha_1+\alpha_2+\alpha_3 = 2$$

$$\Rightarrow 1 + \alpha_2 + (-2) = 2 \Rightarrow \alpha_2 = 3$$

substitute $\alpha_1\, \alpha_2\, \alpha_3$ in eqn (1)

$\therefore$ Hence the soln of recurrence relation is $a_n = (1+3n-2n^2)(-1)^n$

Assignment

(1) Solve $a_n = a_{n-1} + 6a_{n-2}$, for $n \geq 2$, $a_0 = 3$, $a_1 = 6$

(2) $a_n = -4a_{n-1} - 4a_{n-2}$, $a_0 = 0$, $a_1 = 1$

(3) $a_n = a_{n-1} + a_{n-2}$, $a_0 = 0$, $a_1 = 1$

(4) $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, with $a_0 = 2$, $a_1 = 5$, $a_2 = 15$

(5) $a_n = 6a_{n-1} - 9a_{n-2}$; $a_0 = 1$, $a_1 = 6$

Ans (2) $a_n = n \cdot (-2)^{n-1}$

(3) $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

(4) $a_n = 1 - 2^n + 2 \cdot 3^n$ (5) $a_n = (1+n) \cdot 3^n$

Linear Non Homogeneous Recurrence Relation with Constant Coefficients

Suppose, The kth degree Linear Non Homogeneous R.R with constant coefficients is

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n) \quad \text{---(1)}$$

The soln of eqn (1) is

$$\boxed{a_n = a_n^{(h)} + a_n^{(p)}}$$

where $a_n^{(h)}$ = Homogeneous soln

$a_n^{(p)}$ = Particular soln.

Soln of Homogeneous R.R when $F(n)=0$

In previous, we have discussed, The homogeneous soln of Linear R.R.

Method of finding The Particular Soln. when $F(n)\neq 0$

To find Particular Soln of Recurrence Relation, we will consider a Trial soln on the basis nature of $F(n)$.

Case I when $F(n) = b^n$ and b is not a part of characteristic eqn.

Suppose The Trial soln

$$\boxed{a_n^{(p)} = p \cdot b^n}$$

where 'p' is constant.

Case-II When $F(n)=b^n$, and b is a part of C.E.

(i) If multiplicity of b is one.
i.e. b is match only single root of C.E.
Suppose the trial Sol^n is $\boxed{a_n^{(p)} = p \cdot n \cdot b^n}$

(ii) If multiplicity of b is two
i.e. b is match with two root of C.E.
Then suppose the Trial Sol^n is $\boxed{a_n^{(p)} = p n^2 \cdot b^n}$

(iii) Similarly, If multiplicity of b is three.
i.e. b is match with three root of C.E.
Then, suppose the trial Sol^n is $\boxed{a_n^{(p)} = p \cdot n^3 \cdot b^n}$

Case-III When $F(n)$ is a polynomial in 'n'.

(1) If $F(n)$ is a 1st degree poly. in n.
Then Trial Sol^n is $\boxed{a_n^{(p)} = P_0 + P_1 n}$

(2) If $F(n)$ is 2nd degree poly. in 'n'.
Then Trial Sol^n is $\boxed{a_n^{(p)} = P_0 + P_1 n + P_2 n^2}$

(3) If $F(n)$ is a 3rd degree poly. in 'n'.
Then Trial Sol^n is $\boxed{a_n^{(p)} = P_0 + P_1 n + P_2 n^2 + P_3 n^3}$

Note - If $F(n) =$ Constant (K)
and '1' is not a characteristic ~~eqn~~ root of the Homogeneous eqn.
Then assume a particular Sol^n.

$\boxed{a_n^{(p)} = P}$ P: constant

Ex- Solve
$a_n = 3a_{n-1} + 2n$

Soln. Given $a_n = 3a_{n-1} + 2n$ —— (1)

Now, The associated Homogeneous R.R is
$$a_n = 3a_{n-1}$$
∴ The characteristic eqn is
$$r - 3 = 0 \Rightarrow r = 3$$
∴ The ~~sol~~ Homogeneous soln is $a_n^{(h)} = \alpha \cdot 3^n$ —— (2)

Next to find particular soln
Here $F(n) = 2n$ (is a polynomial of degree 1)
So, Then we will assume a particular soln is
$$a_n^{(p)} = p_0 + p_1 n \quad \text{—— (3)}$$
Now putting the value of a_n in eqn (1), we get
$$p_0 + p_1 n = 3(p_0 + p_1(n-1)) + 2n$$
$$\Rightarrow p_0 + p_1 n - 3p_0 + 3p_1 n - 3p_1 + 2n$$
$$\Rightarrow 2p_0 + 2p_1 n - 3p_1 + 2n = 0$$
$$\Rightarrow (2p_0 - 3p_1) + (2p_1 + 2)n = 0$$
It follows That $p_0 + p_1 n$ is a soln
$$\Leftrightarrow 2p_0 - 3p_1 = 0 \text{ and } 2p_1 + 2 = 0$$
$$p_1 = -1$$
$$\Rightarrow 2p_0 + 3 = 0$$
$$\Rightarrow p_0 = -3/2$$
putting the value of p_0 & p_1 in eqn (3)
$$a_n^{(p)} = -3/2 + (-1) \cdot n = -3/2 - n$$
∴ Hence The soln of R.R is
$$a_n = a_n^{(h)} + a_n^{(p)}$$
$$= \alpha \cdot 3^n + (-n - \tfrac{3}{2})$$
(Ans)

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Ex-4 Solve $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

Soln Given R.R is

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n \quad \text{--- (1)}$$

Now the associated Homogeneous eqn is

$$a_n = 5a_{n-1} - 6a_{n-2}$$

$$\Rightarrow a_n - 5a_{n-1} + 6a_{n-2} = 0$$

∴ The Characteristic eqn is

$$r^2 - 5r + 6 = 0$$

$$\Rightarrow r^2 - 3r - 2r + 6 = 0 \Rightarrow (r-3)(r-2) = 0$$

$$\Rightarrow r = 3, 2$$

∴ The Homogeneous soln is

$$a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n \quad \text{--- (2)}$$

where α_1 & α_2 constants.

Next to find particular soln ($a_n^{(p)} = ?$)

Here $F(n) = 7^n$

7 is not a characteristic root of homogeneous eqn

Then, we assume that a trial soln.

$$\boxed{a_n^{(p)} = P \cdot 7^n} \quad \text{--- (3)} \quad \text{where } P \text{ is constant.}$$

Now substitute the value of a_n in eqn (1)

we get

$$P \cdot 7^n = 5 \cdot P \cdot 7^{n-1} - 6P \cdot 7^{n-2} + 7^n$$

$$\Rightarrow P = 5P \cdot 7^{-1} - 6P\,7^{-2} + 1$$

$$\Rightarrow 7^2 P = 7 \cdot 5P - 6P + 7^2$$

$$\Rightarrow 49P = 35P - 6P + 49 \Rightarrow 20P = 49$$

$$\Rightarrow \boxed{P = \frac{49}{20}}$$

Putting the value of 'P' in eqn (3)

we get

$$\boxed{a_n^{(p)} = \frac{49}{20} 7^n}$$

Hence the soln of R.R is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$\Rightarrow \boxed{a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + \frac{49}{20} \cdot 7^n}.$$

(Ans)

Ex-5 Solve, $a_n = a_{n-1} + 7$

Soln given $a_n = a_{n-1} + 7$ —(1)

Now the associated Homogeneous eqn is

$$a_n = a_{n-1}$$

∴ The characteristic eqn is $r - 1 = 0 \Rightarrow r = 1$

∴ The Homogeneous soln is $\boxed{a_n^{(h)} = \alpha_1 \cdot 1^n}$

Next to find particular soln.

Since $F(n) = 7$

we write $F(n) = 7 \cdot 1^n$ $\left(\because 7 = 7 \cdot 1^0 = 7 \cdot 1^1 = 7 \cdot 1^2 \ldots = 7 \cdot 1^n \right)$

∴ 1 is the characteristic root of Homogeneous eqn.

Then, we assume a particular soln.

$$a_n^{(p)} = p \cdot n \cdot 1^n \quad (\because \text{multiplicity of } 1 \text{ is } 1)$$

$$\Rightarrow a_n^{(p)} = p \cdot n \quad \text{—(2)}$$

Now substitute the value of a_n in eqn (1)

we get $pn = p(n-1) + 7$

$$\Rightarrow 0 = -p + 7 \Rightarrow p = 7$$

Substitute the value of 'p' in eqn (2)

∴ Particular soln $\boxed{a_n^{(p)} = 7n}$

Hence the soln of R.R is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

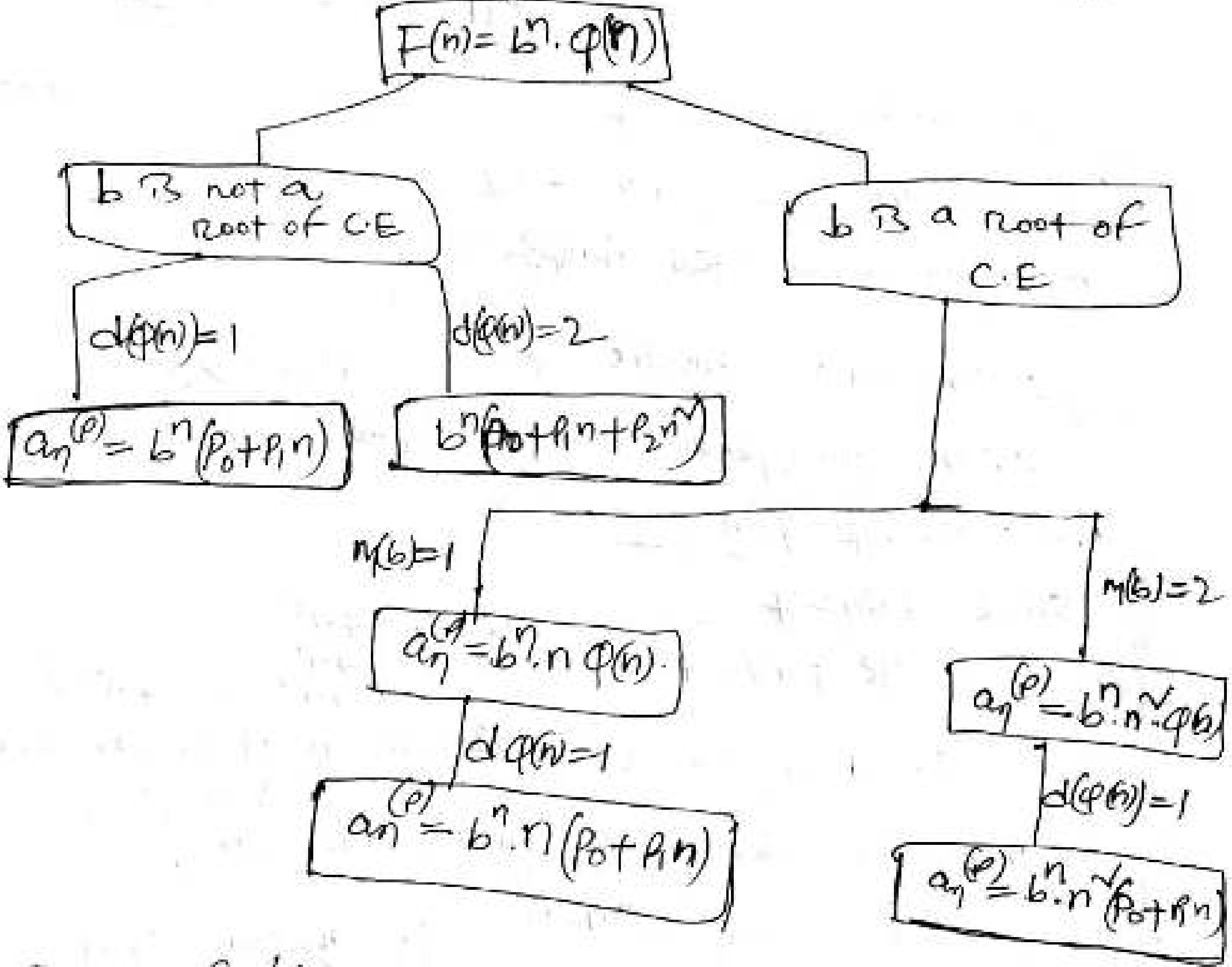
$$\boxed{a_n = \alpha_1 (1)^n + 7n}$$

(Ans)

Case-4 When $F(n) = b^n . \varphi(n)$

where b is constant.

and $\varphi(n)$ is a polynomial in 'n'.



Ex:- Solve

$$a_n = -a_{n-1} + 3n.2^n$$

Soln Given R.R $a_n = -a_{n-1} + 3n.2^n$ ——— (1)

The associated Homogeneous eqn is

$$a_n = -a_{n-1}$$

$$\Rightarrow a_n + a_{n-1} = 0$$

$\therefore$ The Characteristic eqn is

$$r + 1 = 0 \Rightarrow r = -1$$

$\therefore$ The Homogeneous Soln is $\boxed{a_n^{(h)} = \alpha_1 . (-1)^n}$ ——— (2)

Next to find Particular soln.

Since $F(n) = 3n.2^n$

$\therefore$ 2 is not a root of C.E.

and n of degree one.

Then we will assume a particular soln is

$$\boxed{a_n^{(p)} = 2^n(P_0 + P_1 n)} \quad \text{--- (3)}$$

Substitute the value of a_n in eqn (1)

$$2^n(P_0+P_1 n) = -2^{n-1}(P_0+P_1(n-1)) + 3n\cdot 2^n$$

$$\Rightarrow \cancel{2^n}(P_0+P_1 n) = \cancel{2^n}\left(-2^{-1}(P_0+P_1(n-1)) + 3n\right)$$

$$\Rightarrow P_0+P_1 n + \frac{1}{2}(P_0+P_1(n-1)) = 3n$$

$$\Rightarrow 3P_0 + 3P_1 n - P_1 = 6n$$

$$\Rightarrow (3P_0 - P_1) + (3P_1)n = 6n$$

Comparing in both sides we get

$3P_0 - P_1 = 0$, and $3P_1 = 6$

$$\Rightarrow P_1 = 2$$

$\therefore$ $3P_0 - P_1 = 0 \Rightarrow 3P_0 = 2$

$$\Rightarrow P_0 = \frac{2}{3}$$

Substitute P_0 & P_1 in eqn (3)

we get $a_n^{(p)} = 2^n\left(\frac{2}{3} + 2n\right)$

$\therefore$ Hence the soln of R.R is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$\boxed{a_n = a_1\cdot(-1)^n + 2^n\left(\frac{2}{3} + 2n\right)}$$

Ex (1) which are linear Homogeneous and find degree = ?

		Ans	
(1)	$a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$	–	Linear Homogeneous degree = 3
(2)	$a_n = a_{n-1}^2 + a_{n-2}$	X	not defined
(3)	$a_n = a_{n-1} + 2$	X	"
(4)	$a_n = a_{n-2}$	✓	2
(5)	$a_n = 2n a_{n-1} + a_{n-2}$	X	not defined
(6)	$a_n = a_{n-1} + n$	X	"

Case 5: When $F(n) = b^n + \varphi(n)$

where $\varphi(n)$: Polynomial in n

b : constant.

$\boxed{F(n) = b^n + \varphi(n)}$

- b is not a root of C.E
  - $d(\varphi(n)) = 1$: $a_n^{(p)} = b^n + P_0 + P_1 n$
  - $d(\varphi(n)) = 2$: $a_n^{(p)} = b^n + P_0 + P_1 n + P_2 n^2$
- $m(b) = 1$: $a_n^{(p)} = b^n . n + \varphi(n)$
  - $d(\varphi(n)) = 1$: $a_n^{(p)} = b^n . n + (P_0 + P_1 n)$
- $m(b) = 2$: $a_n^{(p)} = b^n . n^2 + \varphi(n)$
  - $d\varphi(n) = 1$: $a_n^{(p)} = b^n . n^2 + P_0 + P_1 n$

Ex- Solve $a_n = 5a_{n-1} - 6a_{n-2} + 2^n + n$ $\quad a_0 = 1$, $a_1 = 1$

Sol^n

Characteristic eq^n is

$r^2 - 5r + 6 = 0$

$\Rightarrow (r-3)(r-2) = 0 \Rightarrow r = 2, 3$

$\therefore$ Homogeneous Sol^n is $a_n^{(h)} = \alpha_1 . 2^n + \alpha_2 . 3^n$.

Since $F(n) = 2^n + n = b^n + \varphi(n)$

$\therefore$ 2 is a root of Homogeneous eq^n. and n is a polynomial of degree 1.

Then we will assume a Particular Sol^n.

$$\boxed{a_n^{(p)} = p b^n n + (p_0 + p_1 n)}$$

Substitute the value of a_n in eqn ① –

(You solve it)

Ans – $a_n = -5 \cdot 2^n + \frac{17}{4} 3^n - n \cdot 2^{n+1} + \frac{n}{2} + \frac{7}{4}$

Assignment –

① Solve, $a_n = 2a_{n-1} + 3^n$

② $a_n = -5a_{n-1} - 6a_{n-2} + 3n^2 - 2n + 1$ — 2nd degree polyn so take $\left(a_n^{(p)} = p_0 + p_1 n + p_2 n^2\right)$

③ $a_n = 2a_{n-1} - a_{n-2} + 7$

④ $a_n = 5a_{n-1} - 6a_{n-2} + 1$

⑤ $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$

Ans – ① $a_n = \alpha \cdot 2^n + 3^{n+1}$

② $a_n = \alpha_1 \cdot (-3)^n + \alpha_2 \cdot (-2)^n + \frac{1}{4} n^2 + \frac{13}{24} n + \frac{71}{288}$

④ $a_n = \alpha_1 (3)^n + \alpha_2 \cdot 2^n + \frac{1}{2}$

③ $a_n = (\alpha_1 + \alpha_2 n) 1^n + \frac{7}{2} n^2$

⑤ $a_n = (\alpha_1 + n\alpha_2) 2^n + n^2 \left(\frac{n}{6} + 1\right) 2^n$

⑥ Solve $a_n + 6a_{n-1} + 9a_{n-2} = 3$ — ①

Hints – Ans $r = -3, -3$

$a_n^{(h)} = (\alpha_1 + \alpha_2 n)(-3)^n$

$F(n) = 3$ (const.) and 1 is not a root of H.R.R.

So, assume $\boxed{a_n^{(p)} = p}$

Substitute the value of a_n in eqn ① to find $p = ?$ and $a_n = a_n^{(h)} + a_n^{(p)} = ?$

Generating Function

Def^n The generating function for the sequence $a_0, a_1, \ldots a_n, \ldots$ of real numbers is the infinite series -

$$G(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

$$= \sum_{k=0}^{\infty} a_k x^k$$

Ex- $\{2^n\} = (2^0, 2^1, 2^2, \ldots\ldots)$

$$G(x) = 2^0 + 2^1x + 2^2 \cdot x^2 + \cdots + 2^n x^n + \cdots$$

$= \sum_{n=0}^{\infty} 2^n x^n$. which can be written closed form $\left(G(x) = (1-2x)^{-1} = \frac{1}{1-2x}\right)$

Ex- Find Generating function for the following sequence.

① $\{1, 1, 1, 1, \ldots\}$

$$G(x) = 1 + x + x^2 + \cdots$$
$$= (1-x)^{-1}$$

② $\{1, -2, 3, -4, \ldots\}$

$$G(x) = 1 - 2x + 3x^2 - 4x^3 + \cdots$$
$$= (1+x)^{-2}$$

③ $\{0, 1, 2, 3 \ldots\}$

$$G(x) = 0 + 1x + 2x^2 + \cdots$$
$$= x(1 + 2x + 3x^2 + \cdots)$$
$$= x(1-x)^{-2}$$

④ $a_n = \{1, 3, 9, 27 \ldots\}$

$$G(x) = 1 + 3x + 9x^2 + 27x^3 + \cdots$$
$$= 1 + 3x + (3x)^2 + (3x)^3 + \cdots$$
$$= (1-3x)^{-1} \quad \left(\because (1-\alpha)^{-1} = 1 + \alpha + \alpha^2 + \alpha^3 + \cdots\right)$$

Some important Results

Generating function $(G(x))$	Numeric function (a_k)
① $G(x)=\sum_{k=0}^{\infty} 1\cdot x^k = 1+x+x^2+\cdots+x^k+\cdots = (1-x)^{-1} = \frac{1}{1-x}$	$a_k=1$
② $G(x)=\sum_{k=0}^{\infty} r\cdot x^k = r+rx+rx^2+\cdots+rx^k+\cdots = r(1-x)^{-1} = \frac{r}{1-x}$	$a_k=r$
③ $G(x)=\sum_{k=0}^{\infty} a^k\cdot x^k = 1+ax+a^2x^2+\cdots+a^kx^k+\cdots = (1-ax)^{-1} = \frac{1}{1-ax}$	$a_k=a^k$ where 'a' is constant.
④ $G(x)=\sum_{k=0}^{\infty} k\cdot x^k = 0+x+2x^2+3x^3+\cdots+kx^k+\cdots = x(1+2x+3x^2+\cdots+kx^{k-1}+\cdots) = x(1-x)^{-2} = \frac{x}{(1-x)^2}$	$a_k=k$
⑤ $G(x)=\sum_{k=0}^{\infty} k^2\cdot x^k = 0+1^2x+2^2x^2+3^2x^3+\cdots+k^2x^k+\cdots = x(x+1)(1-x)^{-3} = \frac{x(x+1)}{(1-x)^3}$	$a_k=k^2$
⑥ $G(x)=\sum_{k=0}^{\infty}(k+1)x^k = 1+2x+3x^2+\cdots+(k+1)x^k+\cdots = (1-x)^{-2} = \frac{1}{(1-x)^2}$	$a_k=k+1$
⑦ $G(x)=1\cdot2\cdot x+2\cdot3x^2+3\cdot4x^3+\cdots+k(k+1)+\cdots = 2x(1-x)^{-3} = \frac{2x}{(1-x)^3}$	$a_k=k(k+1)$
⑧ $G(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!} = 1+x+x^2/2!+x^3/3!+\cdots = e^x$	$a_k=\frac{1}{k!}$

Generating function ($G(x)$)	Numeric function (a_k)
(9) $G(x)=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k}x^k = x-x^2/2+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$	$a_k=\frac{(-1)^{k+1}}{k}$
(10) $G(x)=\sum_{k=0}^{n}x^k = 1+x+x^2+\cdots+x^n = \frac{1-x^{n+1}}{1-x}$	1 if $k\le n$, otherwise

Using Generating function To solve Recurrence Relations

Examples (1) solve the Recurrence Relation

$a_k = 3a_{k-1}$ for $k=1,2,3\cdots$

(using generating function) and $a_0=2$

Sol^n - given R.R is $a_k = 3a_{k-1}$ ——— (1)

$k=1,2,3\cdots$

and $a_0=2$

Let $G(x)$ be the Generating function for the sequence $\{a_k\}$

i.e $G(x) = \sum_{k\ge0}^{\infty} a_k x^k$

$= a_0x^0 + a_1x_1 + a_2x^2 + \cdots + a_kx^k + \cdots$

Now, $a_k = 3a_{k-1}$

Taking x^k multiply in both sides we get

$a_kx^k = 3a_{k-1}x^k$

Taking summation $k=1$ to ∞ in both sides we get

$$\sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} 3a_{k-1} x^k$$

$$= 3\sum_{k=1}^{\infty} a_{k-1} x^k$$

$$\Rightarrow a_1x^1 + a_2x^2 + \cdots\cdots + a_kx^k + \cdots\cdots$$

$$= 3(a_0x^1 + a_1x^2 + a_2x^3 + \cdots)$$

$$\Rightarrow (a_0 + a_1x^1 + a_2x^2 + \cdots\cdots + a_kx^k + \cdots\cdots) - a_0$$

$$= 3x(a_0 + a_1x + a_2x^2 + \cdots)$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k x^k - a_0 = 3x\sum_{k=0}^{\infty} a_k x^k$$

$$\Rightarrow G(x) - a_0 = 3x\,G(x)$$

$$\Rightarrow G(x) - 3x\,G(x) = a_0$$

$$\Rightarrow G(x)(1-3x) = 2$$

$$\Rightarrow \boxed{G(x) = \frac{2}{1-3x}}$$

$$\Rightarrow G(x) = 2\sum_{k=0}^{\infty} 3^k . x^k \quad \left(\because \frac{1}{(1-ax)} = \sum_{k=0}^{\infty} a^k x^k\right)$$

$$\Rightarrow \boxed{G(x) = \sum_{k=0}^{\infty} 2\cdot 3^k \cdot x^k}$$

where $a_k = 2\cdot 3^k$

Example-② Solve R.R using Generating function

$$a_k - 2a_{k-1} - 3a_{k-2} = 0$$

with initial condition

$a_0 = 3 \quad a_1 = 1$

Soln - given R.R is

$$a_k - 2a_{k-1} - 3a_{k-2} = 0$$

Taking x^k multiply in both sides

$$\Rightarrow a_k x^k - 2a_{k-1} x^k - 3a_{k-2} x^k = 0$$

$\Rightarrow$ Taking summation $k=2$ to ∞ in both sides

$$\Rightarrow \underbrace{\sum_{k=2}^{\infty} a_k x^k}_{\text{1st part}} - 2\underbrace{\sum_{k=2}^{\infty} a_{k-1} x^k}_{\text{2nd part}} - 3\underbrace{\sum_{k=2}^{\infty} a_{k-2} x^k}_{\text{3rd part}} = 0 \quad (1)$$

1st part

$$\sum_{k=2}^{\infty} a_k x^k = a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots - a_0 - a_1 x$$

$$= G(x) - a_0 - a_1 x$$

2nd part

$$\sum_{k=2}^{\infty} a_{k-1} x^k = a_1 x^2 + a_2 x^3 + \cdots$$

$$= x(a_1 x + a_2 x^2 + \cdots)$$

$$= x(a_0 + a_1 x + a_2 x^2 + \cdots) - a_0 x$$

$$= x G(x) - a_0 x$$

$$= x(G(x) - a_0)$$

3rd part

$$\sum_{k=2}^{\infty} a_{k-2} x^k = a_0 x^2 + a_1 x^3 + \cdots$$

$$= x^2 (a_0 + a_1 x + \cdots)$$

$$= x^2 G(x)$$

Now substituting 1st part, 2nd part, 3rd part in eqn (1) we get

$$\Rightarrow G(x) - a_0 - a_1 x - 2 \cdot (x (G(x) - a_0) - 3 x^2 G(x) = 0$$

$$\Rightarrow G(x) - 3 - x - 2(x(G(x) - 3) - 3x^2 G(x) = 0$$ Here $a_0 = 3$, $a_1 = 1$

~~$\Rightarrow G(x)(1+2x$~~

$$\Rightarrow G(x) - 3 - x - 2x G(x) + 6x - 3x^2 G(x) = 0$$

$$\Rightarrow G(x)(1 - 2x - 3x^2) = 3 + x - 6x$$

$$\Rightarrow G(x) = \frac{3-5x}{1-2x-3x^2} = \frac{3-5x}{(1-3x)(1+x)}$$

(using partial fraction) $$= \frac{3-5x}{(1-3x)(1+x)}$$

$$= \frac{1}{1-3x} + \frac{2}{1+x}$$

$$= \sum_{k=0}^{\infty} 3^k \cdot x^k + 2 \cdot \sum_{k=0}^{\infty} (-1)^k \cdot x^k$$

$$= \sum_{k=0}^{\infty} 3^k \cdot x^k + \sum_{k=0}^{\infty} 2 \cdot (-1)^k \cdot x^k$$

$$G(x) = \sum_{k=0}^{\infty} \left(3^k + 2\cdot(-1)^k\right) x^k$$

where $a_k = 3^k + 2\cdot(-1)^k$

Assignment

① Use Generating function to solve R.R $a_k = 7a_{k-1}$ with $a_0 = 5$

② $a_k = 5a_{k-1} - 6a_{k-2}$, $a_0 = 6$, $a_1 = 30$

③ $a_k = 8a_{k-1} + 10^{k-1}$ initial conditions $a_1 = 9$.

④ $a_k = 4a_{k-1} - 4a_{k-2} + k^2$, $a_0 = 2$, $a_1 = 5$

Ans- ② $a_k = 18\cdot 3^k - 12\cdot 2^k$

③ $a_k = \frac{1}{2}\left(8^k + 10^k\right)$

④ $a_k = k^2 + 8k + 20 + (6k - 18)2^k$

Principle of Inclusion and Exclusion

For any Three sets A, B, C

(1) $|A \cup B| = |A| + |B| - |A \cap B|$

(2) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

(Addition principle)

Note (1) If $A \cap B = \varphi$, Then $|A \cap B| = 0$

So. $|A \cup B| = |A| + |B|$

(2) $|A \cup B| \leq |A| + |B|$ (3) $|A \cap B| \leq \min(|A|, |B|)$

(4) $|A - B| \geq |A| - |B|$

Suppose For any Four sets A, B, C, D
Then we write union of The 4 sets –

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

In general For a given finite set

$A_1 A_2 \cdots A_n$

The no. of elements in The union

$A_1 \cup A_2 \cup \cdots \cup A_n$ is

$|A_1 \cup A_2 \cup \cdots \cup A_n| =$?

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Examples ① A total of 1232 students have taken a course in spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both spanish and French, 23 have taken courses in both spanish and Russian and 14 have taken courses in both French and Russian, If 2092 students have taken at least one of spanish, French and Russian, how many students have taken a course in all Three languages?

Soln

Let S be the set of students who have taken a course in spanish

F: set of students who have taken a course in French.

R: set of students who have taken a course in russian.

Then $|S| = 1232$, $|F| = 879$, $|R| = 114$,

$|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$

and $|S \cup F \cup R| = 2092$

~~Now~~ $|S \cap F \cap R| = ?$

Now we have

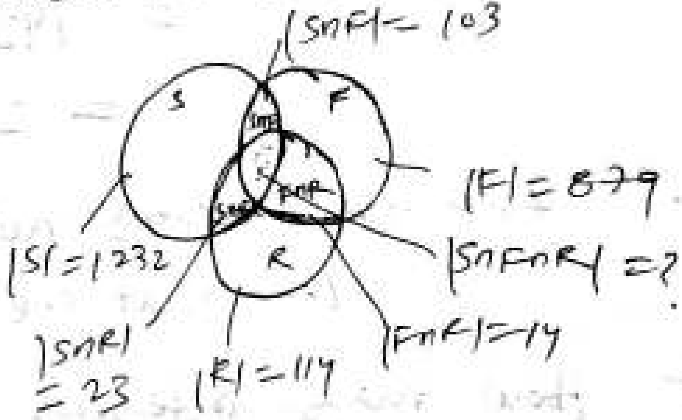
$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$$

$\Rightarrow \quad |S \cap F \cap R| = 7$

$\therefore$ There are 7 students who have taken a course in S, F, and R.

Ven diagram



Example ② How many positive integers not exceeding 1000 are divisible by 7 or 11.

Soln – Let A : be the set of +ve integers not exceeding 1000 that are divisible by 7.

B : be the set of +ve integers not exceeding 1000 that are divisible by 11.

$A \cup B$ is the set of integers not exceeding 1000 that are divisible by 7 or 11.

and $A \cap B$ is the set of integers not exceeding 1000 that are divisible by both 7 and 11.

Now we have $|A \cup B| = |A| + |B| - |A \cap B|$

$$= \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{7 \cdot 11} \right\rfloor$$

$$= 142 + 99 - 12$$

$$= 220$$

$\therefore$ 220 the integers not exceeding 1000 that are divisible by 7 or 11.

Ex- How many integers between 1 and 250 that are divisible by any of integers 2, 3, 5 & 7.

~~Ans = 193~~

Soln Let A_1 : set of integers between 1 & 250 that are divisible by 2.

A_2 : " divisible by 3

A_3 " " " " 5

A_4 : " " " 7

$|A_1| = \left\lfloor \frac{250}{2} \right\rfloor = 125$

$|A_2| = \left\lfloor \frac{250}{3} \right\rfloor = 83$

$|A_3| = \left\lfloor \frac{250}{5} \right\rfloor = 50$, $|A_4| = \left\lfloor \frac{250}{7} \right\rfloor = 35$

$|A_1 \cap A_2| = \left\lfloor \frac{250}{5\times 3} \right\rfloor = 41$, $|A_1 \cap A_3| = \left\lfloor \frac{250}{2\times 5} \right\rfloor = 25$

$|A_1 \cap A_4| = \left\lfloor \frac{250}{2\times 7} \right\rfloor = 17$, $|A_2 \cap A_3| = \left\lfloor \frac{250}{3\times 5} \right\rfloor = 16$

$|A_2 \cap A_4| = \left\lfloor \frac{250}{3\times 7} \right\rfloor = 11$, $|A_3 \cap A_4| = \left\lfloor \frac{250}{5\times 7} \right\rfloor = 7$

$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{250}{2\times 3\times 5} \right\rfloor = 8$, $|A_1 \cap A_3 \cap A_4| = \left\lfloor \frac{250}{2\times 5\times 7} \right\rfloor = 3$

$|A_1 \cap A_2 \cap A_4| = \left\lfloor \frac{250}{2\times 3\times 7} \right\rfloor = 5$, $|A_3 \cap A_3 \cap A_4| = \left\lfloor \frac{250}{3\times 5\times 7} \right\rfloor$
$= 2$

$|A_1 \cap A_2 \cap A_3 \cap A_4| = \left\lfloor \frac{250}{2\times 3\times 5\times 7} \right\rfloor = 1$

we have

$|A_1 \cup A_2 \cup A_3 \cup A_4| = 125 + 83 + 50 + 35 - 41 - 25$
$- 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2$
$= 193 \quad -1$.