

Matrix Algebra

Module-I

Matrix:-

A matrix is a rectangular array of numbers (or functions) enclosed in a bracket. These numbers (or functions) are called ~~all~~ entries or elements of the Matrix.

Example:-

$$\begin{matrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} & , & \begin{bmatrix} 1 \\ 2 \end{bmatrix} & , & \begin{bmatrix} 2 & 3 & 1 \\ 2 & 0 & 0 \end{bmatrix} & , & \begin{bmatrix} 1 & 3 \end{bmatrix} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 2 \times 2 \text{ Matrix} & & \text{column} & & 2 \times 3 & & \text{Row} \\ & & \text{Matrix} & & \text{Matrix} & & \text{Matrix} \end{matrix}$$

General notation:-

A Matrix is denoted by,

$$A = [a_{ij}]_{m \times n}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

consists of m rows and n columns with mn number of elements.

Here a_{ij} is the entry in the i th row and j th column.

If $m=n$, then the matrix A is an $n \times n$ square

Matrix,

Diagonal of the square matrix, $a_{11}, a_{22}, \dots, a_{nn}$

is called the Main Diagonal or Principal Diagonal of A.

A matrix that is not square is called rectangular matrix.

Vectors:-

A vector is a matrix that has only one row, known as a row vector, or only one column known as a column vector.

Transpose:-

The transpose of a $m \times n$ matrix A is denoted by A^T of order $n \times m$ that has i th row of A as its i th column.

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

Example 1

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad A^T = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Symmetric Matrix:-

A square matrix A is said to be symmetric if $A = A^T$.

Skew-Symmetric Matrix:-

A ~~square symmetric~~ matrix A is said to be skew-symmetric if $A^T = -A$.

Two Matrix A and B are said to be equal iff both are of same order and $a_{ij} = b_{ij}$.

Addition:-

Addition is defined only for matrices A and B of same order and is denoted by $A+B$ and is obtained by adding the corresponding entries.

Note:- Matrices of different sizes cannot be added.

Scalar Multiplication:-

The product of any matrix $A = [a_{ij}]_{m \times n}$ and any scalar c is the matrix $CA = [ca_{ij}]_{m \times n}$ obtained by multiplying each entry in A by c .

$$A + (-B) = A - B \quad (\text{Difference of A and B})$$

Properties:-

- $A+B = B+A$ (Commutative)
- $(A+B)+D = A+(B+D)$ (Associative)
- $A+O = A = O+A$ [Additive identity O (Zero matrix) order same as of A]
- $A+(-A) = O = (-A)+A$ [Additive inverse A' of A]
- $c(A+B) = cA + cB$
- $(c+k)A = cA + kA$
- $c(kA) = (ck)A$
- $\mathbb{I}A = A$
- $(A+B)^T = A^T + B^T$
- $(cA)^T = cA^T$

Multiplication of two Matrices:-

The product $C = AB$ of an $m \times n$ matrix $A = [a_{jk}]$ and an $n \times p$ matrix $B = [b_{jk}]$ is defined if and only if $n = n$, that is,

Number of rows of 2nd factor B = Number of columns of 1st factor A ,

and is then defined as the $m \times p$ matrix $C = [c_{jk}]$ with entries,

$$c_{jk} = \sum_{l=1}^n a_{jl} b_{lk} = a_{j1} b_{1k} + a_{j2} b_{2k} + \dots + a_{jn} b_{nk},$$

$$j = 1, \dots, m, \quad k = 1, \dots, p.$$

Properties:-

- Matrix Multiplication is not commutative as $AB \neq BA$ in general.

- $AB = 0$ doesn't imply $A = 0$ or $B = 0$ or $BA = 0$

- $AC = AB$ doesn't imply $C = B$ (even when $A \neq 0$)

- $A(BC) = (AB)C$ (Associativity)

- $(kA)B = k(AB) = A(kB)$

- $(A+B)C = AC + BC$

- $C(A+B) = CA + CB$

- $(AB)^T = B^T A^T$

Special Matrices:-

Triangular Matrices:-

Upper-triangular Matrices are square matrices that can have nonzero entries only on and above the main diagonal, whereas any entry below the diagonal must be 0.

Lower triangular matrices are square matrices that can have nonzero entries on and below the main diagonal, whereas any entry above the diagonal must be 0.

Diagonal Matrices are square matrices that can have non-zero entries only on the main diagonal whereas entries above and lower diagonal must be zero.

Inner product of two vectors, $a = [a_1, a_2, \dots, a_n]$ and $b = [b_1, b_2, \dots, b_n]$ is denoted by $a \cdot b$ and defined as,

$$a \cdot b = [a_1, a_2, \dots, a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a^T b.$$

$$\rightarrow a \cdot b = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

System of Linear Equation :-

Let us consider a linear system of m equations in n unknowns x_1, x_2, \dots, x_n ,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases} \quad \text{--- (1)}$$

where a_{jk} are called as co-efficient of the system, b_i 's are also constants.

If b_i 's are all zero, then (1) is called a homogeneous system. If at least one b_i is not zero, then (1) is

Called a nonhomogeneous system.

A solution of (1) is a set of numbers x_1, x_2, \dots, x_n that satisfies all the m equations.

A solution vector of (1) is a vector X whose components constitute a solution of (1).

If the system (1) is homogeneous, it has at least the trivial solution, $x_1 = 0, x_2 = 0, \dots, x_n = 0$.

Matrix form of Given System:-

The given linear system can be represented in Matrix form as,

$$AX = B$$

where, A = Co-efficient Matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} A$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The Augmented Matrix of the given system denoted by A and is defined as

$$\tilde{A} = [A | B]$$

$$= \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

Elementary Row Operations

(Row-Equivalent Systems)

- Interchange of two rows.
- Addition of a constant multiple of one row to another row.
- Multiplication of a row by a ~~non~~ zero constant c .

A linearly system S_1 row-equivalent to a linear system S_2 if S_1 can be determined from S_2 by finitely many elementary row operations.

Theorem (Row-Equivalent System)

Row equivalent linear system have the same sets of solutions.

Remark 1-

- A linear system (1) is called overdetermined if it has more equations than unknowns.
- A linear system (1) is called determined if $m=n$.
- A linear system (1) is called underdetermined if

the system (I) has fewer equations than unknowns.

- A system (I) is said to be consistent, if (I) has at least one solⁿ.
- Inconsistent if it has no solution.

Gauss Elimination Method :-

Gauss elimination Method based on reduction of given system in to row-equivalent and then backward substitution.

Example 1 -

Solve the linear system,

$$-x_1 + x_2 + 2x_3 = 2$$

$$3x_1 - x_2 + x_3 = 6$$

$$-x_1 + 3x_2 + 4x_3 = 9$$

Solution -

The given system can be written as,

$$AX = B$$

$$\text{where } A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix}$$

The augmented matrix is,

$$\tilde{A} = [A | B]$$

$$\tilde{A} = \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ -1 & 3 & 4 & 9 \end{array} \right]$$

First step:- Elimination of x_1 .

$$R_3 \rightarrow R_3 - R_1$$

$$R_2 \rightarrow R_2 + 3R_1$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 2 & 2 & 2 \end{array} \right]$$

Second step:- Elimination of x_2

$$R_3 \rightarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 0 & -5 & -10 \end{array} \right]$$

The above row equivalent form gave a set of equation as,

$$-x_1 + x_2 + 2x_3 = 2$$

$$2x_2 + 7x_3 = 12$$

$$-5x_3 = -10$$

$$\Rightarrow x_3 = 2 \Rightarrow \boxed{x_3 = 2}$$

$$\text{Now } 2x_2 + 7x_3 = 12$$

$$\Rightarrow 2x_2 + 14 = 12$$

$$\Rightarrow \boxed{x_2 = -1}$$

$$-x_1 + x_2 + 2x_3 = 2$$

$$\Rightarrow -x_1 + (-1) + 2 \times 2 = 2$$

$$\Rightarrow -x_1 - 1 + 4 = 2$$

$$\Rightarrow -x_1 = -1$$

$$\Rightarrow \boxed{x_1 = 1}$$

\therefore The solution of the given system is,

$$x_1 = 1, x_2 = -1, x_3 = 2.$$

Example 1:-

Solve the given system,

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$

$$6x_1 + 15x_2 + 15x_3 - 54x_4 = 27$$

$$12x_1 - 3x_2 - 3x_3 + 24x_4 = 2$$

Solution:-

The given system can be written in matrix form as,

$$AX = B$$

where,

$$A = \begin{bmatrix} 3 & 2 & 2 & -5 \\ 6 & 15 & 15 & -54 \\ 12 & -3 & -3 & 24 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 27 \\ 2 \end{bmatrix}$$

The Augmented Matrix for the given system is,

$$\tilde{A} = [A|B]$$

$$= \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 6 & 15 & 15 & -54 & 27 \\ 12 & -3 & -3 & 24 & 21 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 4R_1$$

$$\approx \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 11 & 11 & -44 & 11 \\ 0 & -11 & -11 & 44 & -11 \end{array} \right]$$

$$R_3 \rightarrow R_3 + R_2$$

$$\approx \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 11 & 11 & -44 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

The above row equivalent form gave a set of equations.

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$

$$11x_2 + 11x_3 - 44x_4 = 11$$

$$\Rightarrow x_2 + x_3 - 4x_4 = 1$$

$$\Rightarrow \boxed{x_2 = 1 - x_3 + 4x_4}$$

Now / $3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$

$$\Rightarrow 3x_1 + 2 - 2x_3 + 8x_4 + 2x_3 - 5x_4 = 8$$

$$\Rightarrow 3x_1 = 6 - 3x_2$$

$$\Rightarrow \boxed{x_1 = 2 - x_2}$$

\therefore The given system has infinite solutions as,

$$x_1 = 2 - k, \quad x_2 = 1 - p + 4k, \quad x_3 = p, \quad x_4 = k,$$

$$(\text{where, } p, k \in \mathbb{R})$$

Example:-

Solve the given system,

$$3x_1 + 2x_2 + x_3 = 3$$

$$2x_1 + x_2 + x_3 = 0$$

$$6x_1 + 2x_2 + 4x_3 = 6$$

Solution:-

The given system can be written as,

$$AX = B$$

$$\text{where } A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

The augmented matrix is,

$$\tilde{A} = [A | B]$$

$$= \begin{bmatrix} 3 & 2 & 1 & 1 & 3 \\ 2 & 1 & 1 & 1 & 0 \\ 6 & 2 & 4 & 1 & 6 \end{bmatrix}$$

$$R_2 \rightarrow -R_2 + R_1$$

$$\approx \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 6 & 2 & 1 & 6 \end{array} \right]$$

$$R_1 \leftrightarrow R_2$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 3 & 2 & 1 & 3 \\ 6 & 2 & 1 & 6 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 3R_1, \quad R_3 \rightarrow R_3 - 6R_1$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & -4 & 1 & -12 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 4R_2$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 0 & 12 \end{array} \right]$$

The above row equivalent form gave a set of equations,

$$x_1 + x_2 = 3$$

$$-x_2 + x_3 = -6$$

$0 = 12$, which is false.

∴ The given system has no solution.

→ At the end of the Gauss Elimination the reduced row equivalent form will have the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ c_{22}x_2 + \dots + c_{2n}x_n &= b_2^* \\ &\vdots \\ K_{rr}x_r + \dots + K_{rn}x_n &= \tilde{b}_r \\ 0 &= \tilde{b}_{r+1} \\ &\vdots \\ 0 &= \tilde{b}_m \end{aligned}$$

where $r \leq m$ (and $a_{11} \neq 0, c_{22} \neq 0, \dots, K_{rr} \neq 0$)

There are three possible cases:-

1// No solution : if $r < m$ and one of the numbers $\tilde{b}_{r+1}, \tilde{b}_{r+2}, \dots, \tilde{b}_m$ is not zero.

2// Unique solution - Precisely One solution. If $r = n$ and $\tilde{b}_{r+1}, \dots, \tilde{b}_m$ if present are zero.

3// Infinitely Many solution - If $r < n$ and $\tilde{b}_{r+1}, \dots, \tilde{b}_m$ if present are 0.

Problems :-

Solve the given system of linear equations,

$$1/ \quad 4x - 5y + 3z = 16$$

$$-x + 2y - 5z = -2$$

$$3x + 6y - z = 7$$

$$\begin{aligned} 2/ \quad x + y - z &= 9 \\ 8y + 6z &= -6 \\ -2x + 4y - 6z &= 40 \end{aligned}$$

$$\begin{aligned} 3/ \quad 4y + 3z &= 8 \\ 2x - z &= 2 \\ 3x + 2y &= 5 \end{aligned}$$

$$\begin{aligned} 4/ \quad 5x + 5y - 10z &= 0 \\ 2w - 3x - 3y + 6z &= 2 \\ 4w + x + y - 2z &= 4 \end{aligned}$$

$$\begin{aligned} 5/ \quad 2w + 3x + y - 11z &= 1 \\ 5w - 2x + 5y - 9z &= 5 \\ w - x + 3y - 3z &= 3 \\ 3w + 4x - 7y + 2z &= -2 \end{aligned}$$

6// Find the value of a for which the given system,

$$\begin{aligned} x + 2y + 3z &= 5 \\ 2x + y + z &= 4 \\ 3x + 3y + az &= 9 \end{aligned}$$

i) has no solution

ii) has unique solution

iii) has infinite solution

Linear Combination

Given n vectors a_1, a_2, \dots, a_n , a linear combination of these vectors is an expression of the form,

$$c_1 a_1 + c_2 a_2 + \dots + c_n a_n,$$

where c_1, c_2, \dots, c_n are any scalars.

Now consider,

$$c_1 a_1 + c_2 a_2 + \dots + c_n a_n = 0$$

$$\text{If } c_1 = 0, c_2 = 0, \dots, c_n = 0$$

then we can say the vectors a_1, a_2, \dots, a_n are linearly independent.

If at least one $c_i \neq 0$ then vectors a_1, a_2, \dots, a_n are linearly dependent.

Example 1:-

The vectors are linearly independent or dependent.

$$a_1 = \begin{bmatrix} 3 & 0 & 2 & 2 \end{bmatrix}$$

$$a_2 = \begin{bmatrix} -6 & 4 & 2 & 5 \end{bmatrix}$$

$$a_3 = \begin{bmatrix} 21 & -2 & 0 & -15 \end{bmatrix}$$

$$\text{Here } 6a_1 - \frac{1}{2}a_2 - a_3 = 0$$

\therefore The vectors are linearly dependent.

Rank of a Matrix :-

The maximum number of linearly independent row vectors of a matrix $A = [a_{jk}]$ is called the rank of A and is denoted by,

$$\text{rank}(A) / \rho(A)$$

\rightarrow Here we will determine rank of a matrix by two methods i.e., Minor Method / Determinant Method

and Echelon form.

Minor Method / Determinant Method

Definition

A non-zero number r is said to be rank of the matrix A , if

- i) there exists atleast a minor of A of order r . (exists means non-zero quantity)
- ii) Every minor of higher order than r is zero.

Example

Evaluate rank of $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$

Solution

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{vmatrix} = 2(-9+8) + 1(-4-0) - 1(0-6)$$

$$= -2 - 4 + 6 = 0$$

$$\therefore \rho(A) \neq 3$$

Minors of order 2

$$M_1 = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6 - 0 = 6 \neq 0$$

$$\therefore \boxed{\rho(A) = 2}$$

Example 1-

Determine rank of $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$

Solution:-

$$|A| = \begin{vmatrix} -1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{vmatrix}$$

$$= -1(28+2) - 2(1+14) + 3(-4+4)$$

$$= -30 - 30$$

$$= -60 \neq 0$$

$$\therefore \rho(A) = 3.$$

Echelon Form of a matrix :-

→ Any rows of all zeros are below any other non-zero rows.

→ Each leading entry of a row is in a column to the right of the leading entry of the row above it.

→ All entries in a column below a leading entry are zero.

Example 1-

$$\begin{bmatrix} 3 & 2 & 0 & 7 & 0 \\ 0 & 4 & 5 & 10 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Echelon Form :-

We say that a matrix is in reduced echelon form if it is in echelon form and additionally,

- 1/ The leading entry in each nonzero row is 1.
- 2/ Each leading 1 is the only nonzero entry in its column.

Uniqueness of Reduced Echelon Form :-

- We can transform any matrix into a matrix in reduced echelon form using elementary row operations.
- However, no matter what sequence of row operations we use, each matrix is row equivalent to one and only one reduced echelon matrix.

Theorem (Row-equivalent Matrices) :-

Row equivalent matrices have the same rank.

Example :-

Find rank of Matrix, $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix}$

Solution :-

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2.$$

Example 1-

Determine rank of $A = \begin{bmatrix} 8 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

Solution 1-

$$A = \begin{bmatrix} 8 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 2R_1$$

$$R_3 \rightarrow R_3 - 7R_1$$

$$\sim \begin{bmatrix} 8 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2/2$$

$$\sim \begin{bmatrix} 8 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

Example 6

Determine rank of

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 3R_1$$

$$R_4 \rightarrow R_4 - 6R_1$$

$$\approx \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\approx \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - R_2$$

$$\approx \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - R_3.$$

$$\approx \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 3.$$

Theorem (Rank in terms of column vectors)

The rank of a matrix A equals the maximum number of linearly independent column vectors of A .

Theorem (Linearly Independent and Dependent)

p vectors x_1, \dots, x_n (with n components each) are linearly independent, if the matrix with row vectors x_1, x_2, \dots, x_p has rank p ; they are linearly dependent if that rank is less than p .

Theorem 2

p vectors with $n < p$ components are always linearly dependent.

Theorem 6
The vector space \mathbb{R}^n consisting of all vectors with n components has dimension n .

Remark 6

1// Row Rank = Column Rank

2// $\rho(A) \leq \min(m, n)$, where m and n are number of rows and columns respectively.

3// $\rho(A) = 0$ if and only if $A = 0$

4// If A square matrix is invertible then its rank is n and converse is also true.

5// $\rho(A^T A) = \rho(A A^T) = \rho(A) = \rho(A^T)$

6// $\rho(I_n) = n$, $I_n \rightarrow$ Identity matrix of order $n \times n$.

7// $\rho(A+B) \leq \rho(A) + \rho(B)$.

Link of rank and types of solution of linear system

Let A and \tilde{A} be the coefficient matrix and augmented matrix for the linear system $AX = B$ with n unknowns and m equations.

\Rightarrow If $\rho(A) = \rho(\tilde{A})$, then solⁿ exist.

i) If $\rho(A) = \rho(\tilde{A}) = n$, then the linear system has unique solution.

ii) If $\rho(A) = \rho(\tilde{A}) = p < n$, then the linear system has infinite solution.

→ If $f(A) \neq f(\tilde{A})$, then system has no solution.

Vector Space

~~A vector space is a set V of vectors such that~~

A nonempty set V of elements v_1, v_2, \dots is called a vector space if in V there are defined two algebraic operations (i.e., vector addition and scalar multiplication) as follows,

Vector Addition -

For v_1, v_2, v_3

1) $v_1 + v_2 \in V$ [closure property]

2) $v_1 + v_2 = v_2 + v_1$ [commutative property]

3) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ [associative property]

4) $v_1 + 0 = v_1 = 0 + v_1$ [Existence of Additive Identity that is the zero vector which is unique]

5) $v_1 + (-v_1) = 0 = (-v_1) + v_1$

[Existence of Additive Inverse $-v_1$ of v_1 which is unique for each element.]

Scalar Multiplication -

For v_1, v_2, v_3 and scalars $a, b \in \mathbb{R}$

1) $cv_1 \in V$

$$7) a(v_1 + v_2) = av_1 + av_2 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Distributive Property}$$

$$8) (a+b)v_1 = av_1 + bv_1$$

$$9) a(bv_1) = (ab)v_1 = a(bv_1) \quad [\text{Associative property}]$$

$$10) 1a = a$$

Examples -

1. Set of Real numbers and complex numbers.
2. Set of real 2×2 matrices.
3. Set of polynomials with real coefficients.

Basis

A linearly independent set in V consisting of a maximum possible number of vectors in V which spans V is called a basis for V .

Span

The set of all linear combination of given vectors a_1, a_2, \dots, a_p with the same number of component is called the span of these vectors.

A Span is a vector space.

Subspace

A subspace of a vector space V , a nonempty subset of V that itself form a vector space with respect to the two algebraic operations of V .

Dimension:-

Dimension of a vector space V is denoted by $\dim(V)$ and is the number of vectors of a basis of V .

Ex:- $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n), x_i \in \mathbb{R}\}$ is a vector space over \mathbb{R} under Vector Addition (component wise) and Scalar Multiplication (to each component)

Orthogonal Basis for \mathbb{R}^n is,

$$B = \left\{ (1, 0, \dots, 0), (0, 1, \dots, 0), (0, 0, 1, \dots, 0), \dots, (0, 0, \dots, 1) \right\}$$

The vectors in B are linearly Independent and each element of \mathbb{R}^n can be written as linear combination of the vectors of B .

$\therefore B \rightarrow$ Basis,

$$\dim(B) = n.$$

\rightarrow The span of the row vectors of a matrix A is called the row space of the matrix A .

\rightarrow The span of the column vectors of a matrix A is called the column space of A .

Task 6
(Go through examples of Vector space, Subspace, Span, Basis, Dimension, Row space and Column space)

Fundamental Theorem for Linear Systems

Theorem (Non Homogeneous System)

- a) Existence - A linear system of m equations in n unknowns x_1, \dots, x_n

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}$$

has solutions iff the coefficient matrix A and the augmented matrix \tilde{A} have the same rank.

- b) Uniqueness - The system has precisely one solution if and only if this common rank r of A and \tilde{A} equals n .

- c) Infinitely many solutions :- If rank r is less than n ,

then system has infinitely solutions. All of these are obtained by determining r suitable unknowns in terms of remaining $n-r$ unknowns to which arbitrary values can be assigned.

- d) Gauss Elimination -

If solutions exists, they can be obtained by Gauss Elimination Method.

Theorem (Homogeneous System)

A homogeneous linear system,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

\vdots

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

always have trivial solution $x_1 = x_2 = \dots = x_n = 0$.

Nontrivial solution exists if $\text{rank}(A) < n$. If $\text{rank } A = r < n$, these solutions together with $x=0$, form a vector space of dimension $n-r$, called the solution space of the homogeneous system.

Any linear combination of vectors from solution space is also a solution of homogeneous system.

Rank-Nullity Theorem

The solution space of (4) is also called null space of A because $Ax=0$ for every x in the solution space. Its dimension is called the nullity of A .

Theorem

$$\text{rank}(A) + \text{Nullity}(A) = \text{Dim}(A).$$

Theorem (Homogeneous linear system with fewer equations than unknowns)

A homogeneous linear system with fewer equations than unknowns always has non-trivial solutions.

The totality of solutions of a nonhomogeneous linear system can now be characterized by Existence theorem.

Theorem 6

If a nonhomogeneous linear system of equations has solutions, then all these solutions are of the form,

$$x = x_0 + x_h,$$

where x_0 is any fixed solution of nonhomogeneous system and x_h runs through all the solutions of the corresponding homogeneous system.

Problems 1

Evaluate rank of the given matrices,

1// $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 9 & 25 & 12 & 3 \end{bmatrix}$

2// $\begin{bmatrix} 3 & -1 & 5 \\ 2 & -4 & 6 \\ 10 & 0 & 14 \end{bmatrix}$

3// $\begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$

4// $\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$

Are the following sets of vectors linearly independent or dependent?

5// $[1, 0, 0], [0, 1, 0], [0, 0, 1]$

6// $[1, 9, 9, 8], [2, 0, 0, 3], [2, 0, 0, 8]$

7// $[2, -4], [1, 9], [3, 5]$

8// $[7, -3, 17], [1, 1, 0], [0, 1, 1]$

Inverse of a Matrix

(Gauss-Jordan Elimination) :-

The inverse of an $n \times n$ matrix is denoted by A^{-1} and is an $n \times n$ matrix such that

$$AA^{-1} = A^{-1}A = I,$$

where I is the $n \times n$ Identity matrix.

Theorem 2

(Existence of the Inverse)

The inverse A^{-1} of an $n \times n$ matrix A exists if and only if $\text{rank } A = n$, ^{hence} if and only if $\det A \neq 0$.
Hence A is nonsingular if $\text{rank } A = n$, and is singular if $\text{rank } A < n$.

If inverse of A exists, then inverse is Unique. (Uniqueness of the inverse).

If A has an inverse, then A is called a nonsingular Matrix. If A has no inverse, then A is called a Singular Matrix.

Example 1

Evaluate inverse of the matrix by Gauss Jordan

Method, $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

Solution

$$[A|I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array} \right]$$

$$R_2 \rightarrow R_2 + 3R_1 \quad R_3 \rightarrow R_3 - R_1$$

$$\approx \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$\approx \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array} \right]$$

$$R_1 \rightarrow -R_1, \quad R_2 \rightarrow R_2/2, \quad R_3 \rightarrow -R_3/5$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array} \right]$$

$$R_1 \rightarrow R_1 + 2R_3, \quad R_2 \rightarrow R_2 - 3.5R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array} \right]$$

$$R_1 \rightarrow R_1 + R_2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array} \right]$$

Here $A^{-1}A = I = AA^{-1}$ (check it)

$$\therefore A^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Method 2

Write $[A | I]$

By using elementary row operations reduce the above matrix into

$$[I | B]$$

Then B is called as inverse of A.

check $AB = BA = I$

Remark By this procedure if one of the row/column is completely 0 then inverse doesn't exist.

Problems

Evaluate inverse of the given matrices..

$$1/ \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \quad 2/ \begin{bmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad 3/ \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}$$