Matrix Eigenvalue Problems:

Module - II

Let $A = [a_{jk}]$ be a given $n \times n$ matrix and consider the vector equation.

$$\boxed{Ax = \lambda x} \text{ ——————— } (1).$$

Here x is an unknown vector and λ an unknown scalar.

Our aim is to determine both x and λ.

A value of λ for which (1) has a solution $x \neq 0$ is called an eigen value or charecteristic value or latent root of the Matrix A.

The corresponding solutions $x \neq 0$ of (1) are called eigen vectors or charecteristic vectors of A corresponding to that eigen value λ.

The set of the eigen values is called Spectrum of A.

The largest of the absolute values of the eigen values of A is called the Spectral radius of A.

The set of all eigen vectors corresponding to an eigenvalue of A, together with zero vector 0 forms a vector space, called the eigen space of A corresponding to ~~this~~ eigen value.

The problem of determining the eigen values and eigenvectors of a Matrix is called an eigenvalue

Problem:

Let's consider eqⁿ ①

$$AX = \lambda X$$

with $A = [a_{jk}]_{n\times n}$ and $X = [x_j]_{n\times 1}$.

$$\therefore \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1(a_{11}-\lambda) + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

In matrix notation,

$$\boxed{(A-\lambda I)X = 0} \quad \text{——} ②$$

The above system has a nontrivial solution if and only if $\det(A-\lambda I)$ is 0.

$$D(\lambda) = \det(A - \lambda I)$$

$$= \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$$\rightarrow \boxed{D(\lambda) = 0} \quad \text{③}$$

$D(\lambda)$ is called characteristic determinant or characteristic polynomial. Equation (3) is called characteristic equation. The characteristic polynomial is of n^{th} degree in λ.

Theorem (Eigen Values)

The eigen values of a square matrix A are the roots of the characteristic equation ③ of A.

Hence, an $n\times n$ matrix has at least one eigen value and at most n numerically different eigen values.

Theorem (Eigen Vectors)

If X is an eigenvector of a matrix A corresponding to an eigenvalue λ, so is KX with any $K \neq 0$

Properties of Eigen Values and Eigenvectors :-

- A square matrix A and its transpose A^T have the same eigen values.
  (Matrices A and A^T will usually have differen eigen vectors)
- The eigenvalues of a diagonal or triangular Matrix are its diagonal elements.
- Determinant of an $n \times n$ matrix A is the product of their eigen values.
- An $n \times n$ matrix is invertible/non-singular if and only if it doesn't have 0 as an eigen value.
- A matrix is singular if 0 is an eigen value of it.
- If matrix A has eigen value λ with corresponding eigenvector X, then for any $K = 1, 2, 3, \ldots$, A^K has eigen values λ^K corresponding to the same same eigenvector X.
- If A is invertible matrix with eigenvalue λ corresponding to eigenvector X, then A^{-1} has eigen-value λ^{-1} corresponding to the same eigen vector X.
- Trace = Sum of the principal diagonal entries of A
  = Sum of the eigen values of A.
- Eigenvectors of a matrix A with distinct eigenvalues are linearly independent.

- rank(A) = number of non-zero eigen values.
- <u>Algebraic Multiplicity</u> (M_λ)

  The <u>algebraic multiplicity</u> of the eigen value is its multiplicity as a root of the charecteristic polynomial, that is the largest integer k.
- Sum of the algebraic multiplicity is the degree of the charecteristic polynomial.
- <u>Geometric Multiplicity</u> (m_λ)

  The maximum number of linearly independent eigen vectors associated with λ, is reffered to as the eigenvalue's geometric multiplicity.
- The geometric multiplicity of λ is also the dimension of the nullspace of $(A-\lambda I)$, also called as nullity of $(A-\lambda I)$, which can be related to the dimension and rank of $(A-\lambda I)$ as.

  $$G.M. = n - \text{rank}(A-\lambda I)$$
- $1 \leq G.M._\lambda \leq A.M_\lambda \leq n$.
- $G.M. \leq A.M.$ [Relation between A.M (Algebraic Multiplicity) and G.M. (Geometric Multiplicity)]
- The defect of λ is denoted by Δ_λ and is defined as $\Delta_\lambda = M_\lambda - m_\lambda$.

Example :

Find eigen values and eigen vectors of the following Matrix, and hence find algebraic and geometric Multiplicity of each eigen values.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Solution :

The given Matrix is a diagonal Matrix, hence, eigenvalues are principal diagonal entries.

$\therefore \lambda = 3, -8, 4.$

Eigen vectors of $\lambda = 3$

$(A - \lambda I) X = 0$

Let $X_1 = [x_1, x_2, x_3]^T$ be the eigen vector of Matrix corresponding to $\lambda = 3$.

$\therefore (A - 3I) X_1 = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -11 x_2 = 0 \;, \; x_3 = 0$

$\Rightarrow x_2 = 0 \;, \; x_3 = 0$

let $x_1 = k$.

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} = K \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for $K = 1$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $X_2 = [x_1 \ x_2 \ x_3]^T$ be the eigen vector corresponding to the eigen value $\lambda = -8$.

$\therefore (A+8I)X_2 = 0$

$$\Rightarrow \begin{bmatrix} 11 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 11x_1 = 0, \ 12x_3 = 0$

$\Rightarrow x_1 = 0, \ x_3 = 0$

Let $x_2 = a$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} = a\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $a=1$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Let $X_3 = [x_1 \ x_2 \ x_3]^T$ be the eigen vector corresponding to the eigen value $\lambda = 4$.

$\therefore (A-4I)X_2 = 0$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 = 0, \ -12x_2 = 0.$

$\Rightarrow x_1 = 0, \ x_2 = 0.$

Let $x_3 = b$.

$$\therefore X_3 = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} = b\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For $b=1$, $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ The eigen values of the given matrix are $3, -8, 4$ and eigen vectors of the given Matrix are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ respectively.

Eigen values	M_λ	m_λ
3	1	1
-8	1	1
4	1	1.

Example :-

Find the eigen values and eigenvectors of the Matrix A and determine Arithemetic Multiplicity and Geometric Multiplicity of each eigen value, where,

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution :-

The charecteristic equation of A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (-2-\lambda)(-\lambda+\lambda^2-12)+2(6+2\lambda)-3(-4+1-\lambda)=0$

$\Rightarrow 2\lambda-2\lambda^2+24+\lambda^2-\lambda^3+12\lambda+12+4\lambda+9+3\lambda=0$

$\Rightarrow -\lambda^3-\lambda^2+21\lambda+45=0$

$\Rightarrow \lambda^3+\lambda^2-21\lambda-45=0$

$\Rightarrow \lambda^3-5\lambda^2+6\lambda^2-30\lambda+9\lambda-45=0$

$\Rightarrow \lambda^2(\lambda-5)+6\lambda(\lambda-5)+9(\lambda-5)=0$

$\Rightarrow (\lambda^2+6\lambda+9)(\lambda-5)=0$

$\Rightarrow \lambda=5,-3,-3$

$\therefore$ The eigen values of A are $5,-3,-3$.

Let eigen vector vector corresponding to the eigen value $\lambda=5$ be $X_1=[x_1\ x_2\ x_3]$.

Now $(A-5I)X_1=0$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now augmented matrix,

$$\left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{array}\right]$$

$R_1 \to R_1+3R_2$

$$\sim \left[\begin{array}{ccc|c} -1 & -10 & -21 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{array}\right]$$

$R_2 \to R_2 + 2R_1$; $R_3 \to R_3 - R_1$

$$\approx \left[\begin{array}{ccc|c} -1 & -10 & -21 & 0 \\ 0 & -24 & -48 & 0 \\ 0 & 8 & 16 & 0 \end{array}\right]$$

$R_2 \to R_2/(-24)$, $R_3 \to R_3/8$.

$$\approx \left[\begin{array}{ccc|c} -1 & -10 & -21 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\approx \left[\begin{array}{ccc|c} -1 & -10 & -21 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The above row equivalent form can be written as,

$$-x_1 - 10x_2 - 21x_3 = 0$$
$$x_2 + 2x_3 = 0$$

Let $x_3 = K$, then $\boxed{x_2 = -2K}$

Now $-x_1 - 10x_2 - 21x_3 = 0$

$\Rightarrow -x_1 + 20K - 21K = 0$

$\Rightarrow -x_1 - K = 0$

$\Rightarrow \boxed{x_1 = -K}$

$\therefore X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -K \\ -2K \\ K \end{bmatrix} = K\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

For $K = -1$, $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Let eigen vector corresponding to the eigen value $\lambda = -3$ be $X_2 = [x_1 \ x_2 \ x_3]^T$.

Now $(A+3I)X_2 = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented Matrix is,

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$$\approx \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \quad x_1 + 2x_2 - 3x_3 = 0$

$\Rightarrow x_1 = 3x_3 - 2x_2$

Let $x_2 = K$ and $x_3 = P$

$\Rightarrow \boxed{x_1 = 3P - 2K}$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3P - 2K \\ K \\ P \end{bmatrix} = P\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + K\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ The eigen vectors corresponding to the eigen values, $\lambda = -3$ are $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

λ	M_λ	m_λ
-3	2	2
5	1	1

Example :-

Find trace of A, where

i) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 7 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 2 & 4 \end{bmatrix}$

Solⁿ :-

i) $\text{Trace}(A) = 2+4+7 = 13$

ii) $\text{Trace}(A) = 1+2+4 = 7$

Problems :-

Find trace and determinant of given matrices. Determine eigen values and eigen vectors also here obtain algebraic and geometric Mutiplicity of each eigen values of A.

1// $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ 2// $\begin{bmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{bmatrix}$

3// $\begin{bmatrix} -5 & -2 \\ 2 & -2 \end{bmatrix}$ 4// $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

5// $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 3 \\ 6 & -6 & -6 \end{bmatrix}$ 6// $\begin{bmatrix} 0 & 1 & 2 \\ -4 & 1 & 4 \\ -5 & 1 & 7 \end{bmatrix}$

7// $\begin{bmatrix} 4 & 1 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{bmatrix}$ 8// $\begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 2 \\ 1 & 4 & 4 \end{bmatrix}$

9// $\begin{bmatrix} 1 & 9 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 5 & 3 & 1 & 1 \\ 4 & 7 & 2 & 2 \end{bmatrix}$ 10// $\begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 2 & 1 & 3 \\ 2 & 1 & 6 & 5 \\ 1 & 2 & 4 & 8 \end{bmatrix}$

Symmetric Matrix :-

A real square Matrix $A = [a_{ij}]$, is called symmetric if transposition leaves it unchanged,

$$\boxed{A^T = A,}$$

thus $a_{ij} = a_{ji}$

Skew-Symmetric Matrix :-

A real square Matrix $A = [a_{ij}]$ is called skew-symmetric if transposition gives the negative of A,

$$\boxed{A^T = -A}$$

thus $a_{ij} = -a_{ji}$

→ Any square Matrix A may be written as the ~~sym~~ sum of a symmetric Matrix ~~and~~ R and skew-symmetric Matrix S, where :

$$R = \left(\frac{A + A^T}{2}\right) \quad \text{and} \quad S = \frac{A - A^T}{2}$$

Orthogonal Matrix :-

A real square matrix $A = [a_{ij}]$ is called orthogonal if transposition gives the inverse of A,

$$\boxed{A^T = A^{-1}}$$

Examples:-

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & 2 \\ 5 & 2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} \qquad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Symmetric, Skew-Symmetric, Orthogonal

Properties of Symmetric Matrix :-

1. Sum and difference of two symmetric matrices is again symmetric.
2. This is not always true for the product. Given symmetric matrices A and B, then AB is symmetric if and only if A and B commute i.e, AB = BA.
3. For integer n, A^n is symmetric if A is symmetric.
4. If A^{-1} exists, it is symmetric if and only if A is symmetric.
5. The eigen values of real symmetric Matrix are real.
6. Eigenvectors corresponding to distinct eigen values are orthogonal.

Properties of Skew-Symmetric Matrix :-

1. The sum of two skew-symmetric matrices is skew symmetric
2. A scalar multiple of skew-symmetric matrix is a skew-symmetric.
3. The elements on the principal diagonal of a skew-symmetric Matrix are zero and therefore its trace is 0.

4. Eigen values of skew-symmetric Matrix are purely imaginary or zero.

Orthogonality of Vectors :-

Two vectors a and b is said to be orthogonal if

$$a \cdot b = a^T b = \begin{cases} 0, & \text{if } j \neq K \\ \neq 0, & \text{if } j = K. \end{cases}$$

Orthonormality of column and row Vectors :-

A real square Matrix is orthogonal if and only if its column vectors $a_1, \ldots, a_n$ (and also its row vectors) form an orthonormal system that is,

$$a_i \cdot a_j = a_i^T a_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Properties of Orthogonal Matrix :-

1. The determinant of an orthogonal Matrix has the value +1 or -1

Proof :-

$$1 = \det(I) = \det(AA^{-1})$$

$$= \det(AA^T) \quad [\text{Since } A \text{ is orthogonal}]$$

$$= \det(A)\det(A^T)$$

$$= \det(A)\det(A)$$

$$= [\det(A)]^2$$

$$\Rightarrow \det(A) = +1 \text{ or } -1.$$

2. The eigen values of an orthogonal Matrix A are real or complex conjugates in pairs and have absolute value 1.

5. The product of orthogonal matrices is also orthogonal

4. Any orthogonal Matrix is invertible.

Hermitian Matrix :-

A complex square matrix $A = [a_{ij}]$ is called Hermitian

if $\boxed{\bar{A}^T = A}$,

i.e, $\bar{a}_{ji} = a_{ij}$

Example :-

$$\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1+i & 1-i \\ 1-i & 2 & 2-i \\ 1+i & 2+i & 3 \end{bmatrix}$$

Properies :-

1. The ~~propertie~~ principal diagonal entries of hermitian Matrix are real.

2. If A is Hermitian $A+\bar{A}^T$, $A\bar{A}^T$ and $\bar{A}^T A$ are Hermitian.

3. If A is Hermitian, then A^k is Hermitian for all $k = 1, 2, 3 \ldots$. If A is nonsingular then A^{-1} is Hermitian

4. If A, B are Hermitian, then $aA + bB$ is Hermitian for all real scalars a, b.

5. If a Hermitian matrix is real, then it is a symmetric Matrix.

6. Eigen values of Hermitian matrix are real.

Skew Hermitian Matrix

A complex square matrix $A = [a_{ij}]$ is called Skew-Hermitian if

$$\boxed{\overline{A}^T = -A}$$

i.e, $\overline{a_{ji}} = -a_{ij}$

If a skew Hermitian matrix is real, then it is a skewsymmetric Matrix.

Example :-

$$\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$$

Properties :-

1. The eigen values of skew-Hermitian Matrix are all purely imaginary or 0.

2. All entries on the main diagonal of a Skew-Hermitian matrix have to be purely imaginary or zero.

3. If A and B are skew-Hermitian, then $aA + bB$ is Skew-Hermitian for all real scalars a and b.

4. A is Skew Hermitian iff iA (or $-iA$) is Hermitian.

5. A is Skew Hermitian if and only if the Real part of A is skew symmetric and imaginary part of A is symmetric.

6. If A is skew Hermitian, then A^k is Hermitian if k is an even integer and skew Hermitian if k is an odd integer.

Unitary Matrix :

A complex ~~s~~square matrix $A = [a_{ij}]$ is called Unitary Matrix if $\boxed{\bar{A}^T = A^{-1}}$

Example :

$$\begin{bmatrix} \frac{1}{2} i & \frac{1}{2}\sqrt{3} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{bmatrix}$$

Properties :

1. Eigen values of unitary matrix have absolute value 1.
2. If a ~~[illegible]~~ unitary matrix A is real then it is a orthogonal matrix.
3. Unitary matrix is also known as Normal Matrix
4. If A and B are normal with $AB = BA$ then both AB and $A+B$ are normal
5. The determinant of a unitary matrix has absolute value 1.

Proof : $1 = \det(I) = \det(A\bar{A}^{\dagger})$

$= \det(A\bar{A}^{\dagger}) = \det(A)\det(A^{\dagger})$

$= \det(A)\overline{(\det A)}$

$= |\det(A)|^2$

$\Rightarrow |\det(A)| = 1$

6. Columns of unitary matrix form an orthonormal basis of $\mathbb{C}^n$ with respect to usual inner product.

Similarity of Matrices :-

An $n \times n$ matrix $\hat{A}$ is called similar to an $n \times n$ matrix A if

$$\boxed{\hat{A} = P^{-1} A P}$$

for some nonsingular $n \times n$ matrix P.

This transformation, which gives $\hat{A}$ from A, is called a similarity transformation.

Theorem :-

Eigenvalues and Eigenvectors of Similar Matrices :-

If $\hat{A}$ is similar to A, then $\hat{A}$ has the same eigenvalues of A.

Furthermore, if x is an eigenvector of A, then $y = P^{-1}x$ is an eigenvector of $\hat{A}$ corresponding to the same eigen value.

Theorem :-

(Linear Independence of Eigenvectors)

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigenvalues of $n \times n$ matrix. Then corresponding eigenvectors $x_1, x_2, \ldots, x_n$ form a linearly independent set.

Theorem :-

If an $n \times n$ matrix A has n distinct eigenvalues, then A has a basis of eigenvectors for C^n (or R^n)

Example :-

The matrix $A = \begin{bmatrix} 5 & 3 \\ 3 & -5 \end{bmatrix}$ has a basis of eigen vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponding to the eigenvalues $\lambda_1 = 8$ and $\lambda_2 = 2$.

Theorem :-
(Basis of Eigenvectors)

A Hermitian, SkewHermitian or Unitary matrix has a basis of eigenvectors for C^n that is a unitary system. A symmetric matrix has an orthogonal basis of eigenvectors for R^n.

Diagonalization :-

If an $n \times n$ Matrix A has a basis of eigen vectors, then,

$$D = X^{-1} A X,$$

is diagonal, with the eigenvalues of A as the entries on the Main diagonal. Here X is the Matrix with these eigenvectors as column vectors.

Powers of A :-

$$D^2 = (X^{-1} A X)(X^{-1} A X)$$
$$= X^{-1} A(X X^{-1}) A X$$
$$= X^{-1} (A I A) X$$
$$= X^{-1} A^2 X$$

Multiply X on left side and X^{-1} right right of both sides of equation

$$X D^2 X^{-1} = X X^{-1} A^2 X X^{-1}$$

$$\Rightarrow \boxed{X D^2 X^{-1} = A^2}$$

Similarly $\boxed{A^3 = X D^3 X^{-1}}$

Also $\boxed{A^m = X D^m X^{-1}}$

Method to Diagonalize a Matrix :-

1. Denote the given matrix by A.
2. Find eigen values $\lambda_1, \lambda_2, ---, \lambda_n$ that are distinct of A.
3. Find eigen vectors $X_1, X_2, ---, X_n$ of $\lambda_1, \lambda_2, ---, \lambda_n$ respectively.
4. Now find a nonsingular matrix by arranging each eigen vectors as a column vector of a matrix.

$$X = [X_1 \;\; X_2 \;\; --- \;\; X_n]$$

5. Find X^{-1}.
6. Determine $D = X^{-1} A X$,

where D is the diagonal matrix with principal diagonal entries as eigen values.

Example :

Diagonalize the given matrix,

$$A = \begin{bmatrix} -1 & -2 & 2 \\ 4 & 3 & -4 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution :

The Charecteristic equation is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & -2 & 2 \\ 4 & 3-\lambda & -4 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda) \begin{vmatrix} 3-\lambda & -4 \\ -2 & 1-\lambda \end{vmatrix} - 4 \begin{vmatrix} -2 & 2 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1+\lambda)(\lambda^2 - 4\lambda + 3 - 8) + 4(-2 + 2\lambda + 4) = 0$$

$$\Rightarrow (1+\lambda)(\lambda^2 - 4\lambda + 5) + 8(\lambda + 1) = 0$$

$$\Rightarrow (1+\lambda)(\lambda^2 - 4\lambda + 3) = 0$$

$$\Rightarrow (1+\lambda)(\lambda - 1)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 1, -1, 3$$

Let $X_1 = [x_1, x_2, x_3]^T$ be the eigen vector corresponding to $\lambda = 1$

$$\therefore (A - I)X = 0$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ 4 & 2 & -4 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, $\begin{bmatrix} -2 & -2 & 2 \\ 4 & 2 & -4 \\ 0 & -2 & 0 \end{bmatrix}$

$R_1 \to R_1/2$, $R_2 \to R_2/2$, $R_3 \to R_3/2$

$$\approx \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -2 \\ 0 & -1 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$$\approx \begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$\therefore$ The reduced equivalent form given equation,

$$-x_1 - x_2 + x_3 = 0$$

$$\boxed{x_2 = 0}$$

$$\therefore -x_1 + x_3 = 0$$

$$\Rightarrow x_1 = x_3 = K \text{ (say.)}$$

$$\therefore X_1 = \begin{bmatrix} K \\ 0 \\ K \end{bmatrix} = K \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For $K = 1$,

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let X_2 be the eigen vector corresponding to $\lambda = -1$.

$$\therefore (A + I)X_2 = 0$$

$$\Rightarrow \begin{bmatrix} 0 & -2 & 2 \\ 4 & 4 & -4 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad -x_2 + x_3 = 0 \Rightarrow \boxed{x_2 = x_3}$

$x_1 + x_2 - x_3 = 0 \Rightarrow \boxed{x_1 = 0}$

Let $x_2 = K$

$$\therefore \quad X_2 = \begin{bmatrix} 0 \\ K \\ K \end{bmatrix} = K \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For $K = 1$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Let $X_3 = [x_1 \ \ x_2 \ \ x_3]^T$ be the eigen vector corresponding to $\lambda = 3$

$$\therefore \quad (A - 3I) X_3 = 0$$

$$\Rightarrow \begin{bmatrix} -4 & -2 & 2 \\ 4 & 0 & -4 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Now, $\begin{bmatrix} -4 & -2 & 2 \\ 4 & 0 & -4 \\ 0 & -2 & -2 \end{bmatrix}$

$R_2 \to R_2 + R_1$

$$\approx \begin{bmatrix} -4 & -2 & 2 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$$

$\therefore \quad -4x_1 - 2x_2 + 2x_3 = 0$

$-2x_2 - 2x_3 = 0$

$\Rightarrow \boxed{x_2 = -x_3}$

Now, $-4x_1 - 2x_2 + 2x_3 = 0$

$$\Rightarrow -4x_1 + 2x_3 + 2x_3 = 0$$

$$\Rightarrow -4x_1 + 4x_3 = 0$$

$$\Rightarrow \boxed{x_1 = x_3}$$

$\therefore$ Let $x_3 = k$

$$\therefore X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

For $k = 1$,

$$X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Now the nonsingular matrix,

$$X = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

For Inverse by Gauss Jordan method,

$$[A \mid I]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$R_1 \to R_1 - R_3$, $R_2 \to R_2 + R_3$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

~~Let X~~

$$\therefore \quad X^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix} \qquad [XX^{-1} = X^{-1}X = I]$$

$$\text{Now, } X^{-1}A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 & 2 \\ 4 & 3 & -4 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ -3 & -3 & 3 \end{bmatrix}$$

$\therefore D = X^T A X$

$$= \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ -3 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Transformation of Quadratic form to Principal Axes :-

Let us consider a quadratic form,

$$\boxed{Q = X^T A X} \quad \text{--- (1)}$$

where A is a real symmetric Matrix and X is a column vector. Then A has an orthonormal basis of n eigen vectors. Hence the matrix P with these vectors as column vectors is orthogonal.

So, $P^{-1} = P^T$.

$\therefore \exists$ a diagonal Matrix D, such that

$$D = P^{-1} A P$$

$$\Rightarrow A = P D P^{-1}$$

$$\Rightarrow \boxed{A = P D P^T} \quad \text{--- (2)}$$

On using eq^n (2) in eq^n (1) we will get,

$$Q = X^T P D P^T X$$

$$= (P^T X)^T D P^T X$$

Let $P^T X = Y$, then $\boxed{X = PY}$ (since, $P^T = P^{-1}$) — (3)

$$\therefore \boxed{\begin{aligned} Q &= Y^T D Y \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \end{aligned}} \quad \text{— (4)}$$

Theorem :-

Principal Axes Theorem :-

The sustitution (3) transforms a quadratic form

$$Q = X^T A X = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

to the principal axes form (4) where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigen values of the matrix A, and P is the orthogonal matrix with corresponding eigen vectors $X_1, X_2, \cdots, X_n$ respectively as column vectors.

Example :-

Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the cannonical form and specify the matrix of tranformation.

Solution :-

$$Q = 3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 3 & -2 & 0 \\ 0 & 5 & -2 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Let $B = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 5 & -2 \\ 2 & 0 & 3 \end{bmatrix}$. $A = \dfrac{B + B^T}{2} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$$Q = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

∴ The charecteristic equation of A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(15 - 8\lambda + \lambda^2 - 1) + 1(-3 + \lambda + 1) + 1(1 - 5 + \lambda) = 0$$

$$\Rightarrow (3-\lambda)(\lambda^2 - 8\lambda + 14) - 2 + \lambda + \lambda - 4 = 0$$

$$\Rightarrow 3\lambda^2 - 24\lambda + 42 - \lambda^3 + 8\lambda^2 - 14\lambda + 2\lambda - 6 = 0$$

$$\Rightarrow -\lambda^3 + 11\lambda^2 - 36\lambda + 36 = 0$$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 - 9\lambda^2 + 18\lambda + 18\lambda - 36 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda^2 - 9\lambda + 18) = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 3)(\lambda - 6) = 0$$

$$\Rightarrow \lambda = 2, 3, 6$$

∴ The cannonical form is -

$$\boxed{Q = 2y_1^2 + 3y_2^2 + 6y_3^2}$$

Let $X_1 = [x_1 \; y_1 \; z_1]$ be the eigen vector corresponding to $\lambda = 2$

$(A - 2I) X_1 = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving we get,

$y_1 = 0, \quad z_1 = -x_1$

Let $x_1 = K$

$$\therefore X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} K \\ 0 \\ -K \end{bmatrix} = K \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For $K = 1$,

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Let $X_2 = [x_2 \; y_2 \; z_2]$ be the eigen vector corresponding to $\lambda = 3$.

$(A - 3I) X_2 = 0$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving we get,

$x_2 = y_2, \quad z_2 = y_2$.

Let $y_2 = K$,

$$\therefore X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} K \\ K \\ K \end{bmatrix} = K\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $K=1$,

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $X_3 = [x_3 \ y_3 \ z_3]$ be the eigen vector corresponding to the eigen value $\lambda = 6$.

$$(A - 6I)X_3 = 0$$

$$\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving we will get,

$y_3 = -2z_3$, $x_3 = z_3$. Let $z_3 = K$

$$\therefore X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} K \\ -2K \\ K \end{bmatrix} = K\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

For $K=1$,

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$|X_1| = \sqrt{2}$, $|X_2| = \sqrt{3}$, $|X_3| = \sqrt{6}$

Let $\hat{X}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ $\hat{X}_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ $\hat{X}_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

∴ The orthogonal Matrix,

$$P = [\hat{X}_1 \quad \hat{X}_2 \quad \hat{X}_3]$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

∴ The cannonical form is,

$$Q = Y^T D Y \quad , \text{ where } Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 2y_1^2 + 3y_2^2 + 6y_3^2$$

under the orthogonal transformation

$$Y = P^T X$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{2}} - \frac{z}{\sqrt{2}} \\ \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} \\ \frac{x}{\sqrt{6}} - \frac{2y}{\sqrt{6}} + \frac{z}{\sqrt{6}} \end{bmatrix}$$

$$\Rightarrow y_1 = \frac{x}{\sqrt{2}} - \frac{z}{\sqrt{2}} \; , \; y_2 = \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} \; , \; y_3 = \frac{x}{\sqrt{6}} - \frac{2y}{\sqrt{6}} + \frac{z}{\sqrt{6}}$$

Remark :-

1. The principal axer form is also known as Cannonical form (conic section)

2. Cannonical form is any form which can be expressed as sum of squares of variables.

Problems :-

Diagonalize the given Matrices.

1. $\begin{bmatrix} 16 & 0 & 0 \\ 48 & -8 & 0 \\ 84 & -24 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 1 & -1 \\ -1 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & -2 \\ 2 & 2 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 1 & -1 \\ -4 & 5 & -2 \\ -2 & 1 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 1 & -1 \\ 4 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{bmatrix}$

Transformation of Quadratic Forms to Principal Axes

7. $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$

8. $4x_1^2 + 12x_1x_2 + 13x_2^2 = 16$

9. $17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$

10. $9x_1^2 - 6x_1x_2 + x_2^2 = 40$