Module - III

Vector Differential Calculus

A vector is a quantity that is determined by both its magnitude and direction. Thus it is an arrow or directed line segment. A vector has a tail called its initial point and a tip called its terminal point.

Components of a Vector :-

If a given vector v has initial point $P:(x_1, y_1, z_1)$ and the terminal point $Q:(x_2, y_2, z_2)$ the three numbers

$$\boxed{v_1 = x_2 - x_1, \quad v_2 = y_2 - y_1, \quad v_3 = z_3 - z_1},$$

are called components of v and we write simply,

$$V = [v_1, v_2, v_3].$$

Length of the vector $\underline{V}$:-

$|V|$ = Distance betn initial point and terminal point.

$$\Rightarrow \boxed{|V| = \sqrt{v_1^2 + v_2^2 + v_3^2}}$$

Vector Addition and Scalar Multiplication :-

Two vectors $V = [v_1, v_2, v_3]$ and $W = [w_1, w_2, w_3]$ are added as follows,

$$\boxed{V + W = [v_1 + w_1, v_2 + w_2, v_3 + w_3]}$$

A scalar C multiplied to the vector V as,

$$\boxed{CV = [cv_1, cv_2, cv_3]}$$

Basic Properties of Vector addition and Scalar Multiplication

(a) $V + W = W + V$ (Commutativity)

(b) $(V + W) + P = V + (W + P)$ (Associativity)

(c) $(V + 0) = V = (0 + V)$ [Existence of Additive Identity]

(d) $V + (-V) = 0$ [Existence of Additive Inverse of V (i.e., $-V$) which is unique]

(e) $c(V + W) = cV + cW$

(f) $(V + W)a = Va + Wa$

(g) $(a + b)V = aV + bV$

(h) $1V = V$

(i) $0V = 0$

(j) $(-1)V = -V$.

Here, V, W and P are vectors and a, b, c are scalars. A vector having unit length is called as unit vector.

A vector can also be represented as,

$$\boxed{V = [V_1, V_2, V_3] = V_1 \hat{i} + V_2 j + V_3 K},$$

where i, j and K are the unit vectors along the direction X-axis, Y-axis and Z-axis respectively.

In component form

$i = [1, 0, 0]$, $j = [0, 1, 0]$, $K = [0, 0, 1]$

Inner Product (Dot Product) :

The inner product of two vectors $V = [v_1, v_2, v_3]$ and $W = [W_1, W_2, W_3]$ is denoted by $V \cdot W$ and is defined by,

$$\boxed{V \cdot W = |V||W| \cos\theta},$$

where θ is the angle betn V and W.

$$\cos\theta = \frac{V \cdot W}{|V||W|}$$

$$\Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{V \cdot W}{|V||W|}\right)}$$

In components the inner product of V and W also defined by,

$$\boxed{V \cdot W = v_1 w_1 + v_2 w_2 + v_3 w_3}$$

Orthogonality :

The inner product two nonzero vector is zero iff these vectors are perpendicular.

i, j and K are orthogonal to each other.

$$i \cdot j = j \cdot K = K \cdot i = 0$$

$$i \cdot i = j \cdot j = K \cdot K = 1$$

Properties :

a) $(aU + bV) \cdot W = aU \cdot W + bV \cdot W$ (Linearity)

b) $U \cdot V = V \cdot U$. (commutativity)

c) $U \cdot U \geq 0$, $U \cdot U = 0$ iff $U = 0$. (positivity)

d) $(U+V) \cdot W = U \cdot W + V \cdot W$ (Distributive)

e) $|u \cdot v| \le |u||v|$ (Cauchy-Schwarz Inequality)

f) $|a+b| \le |a| + |b|$ (Triangle Inequality)

Application of Inner Product :-

→ Work done by a force p on a body giving displacement d is $\boxed{p \cdot d}$.

→ Component of a force p in a given direction of a vector $d (\neq 0)$ is $\frac{p \cdot d}{|d|}$.

Vector Product (Cross Product) :-

The vector product of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is denoted by $u \times v$ and defined as,

$$\boxed{u \times v = |u||v| \sin\theta},$$

where θ is the angle betn u and v.

The direction of $u \times v$ is perpendicular to both u and v.

In components the cross product of u and v is

$$u \times v = [u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1]$$

$$= \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$i \times i = j \times j = K \times K = 0$

$i \times j = K$, $j \times i = -K$

$j \times K = i$; $K \times j = -i$

$K \times i = j$; $i \times K = -j$

Properties :-

a) $(aU) \times V = a(U \times V) = U \times (aV)$

b) $U \times (V + W) = U \times V + U \times W$ (Distributive)

c) $U \times V = - V \times U$ (Anticommutative)

d) $U \times (V \times W) \neq (U \times V) \times W$ (Not Associative)

Application of Vector Product :-

Moment of a Force :-

Let a force p acts on a line through a point A Moment vector about a point Q is,

$$m = r \times p,$$

where r is the vector with initial point Q and terminal point A.

Velocity of a Rotating Body :-

Velocity of a rotating B rotating with angular velocity w is denoted by v and defined as,

$$v = w \times r,$$

where r is the position vector of any point on B reffered to a coordinate system with origin

O on the axis of rotation :

Scalar Triple Product :

The scalar triple of three vectors $U = [u_1, u_2, u_3]$ $V = [v_1, v_2, v_3]$ and $W = [w_1, w_2, w_3]$ is denoted by $(U\ V\ W)$ and defined by,

$$(U V W) = U \cdot (V \times W)$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Property : $U \cdot (V \times W) = (U \times V) \cdot W$

Linearly Independence of three vectors :

Three vectors form linealy independent iff their scalar triple product is not zero.

Application :-

→ $|U \times V|$ is the area of a parallelogram with U and V as the adjacent sides.

→ $|(U\ V\ W)|$ is the volume of the parallelopiped with U, V and W as the concurrent edges.

→ The volume of the tetrahedron is $1/6$ of the volume of the parallelopiped.

→ The area of a triangle is $\frac{1}{2}$ of the area of the parallelopiped.

Vector and Scalar Functions and Fields :-

A ~~fraction~~ function is a function whose values are vectors,

$$V = V(P) = [V_1(P), V_2(P), V_3(P)],$$

depending on the points P in space.

A Scalar function is a function whose values are scalars $f = f(P)$, depending on P.

A vector function defines vector field and a scalar function defines scalar field

Limit :-

A vector function $v(t)$ of a real variable t is said to have the limit l as t approaches to, if $V(t)$ is defined in some neighbourhood at t_0 and

$$\lim_{t \to t_0} |V(t) - l| = 0$$

i.e., $\lim_{t \to t_0} |V(t)| = l$.

Continuity :-

A vector function $V(t)$ is said to be continuous at $t = t_0$ if it is defined in some neighborhood at $t = t_0$ and

$$\lim_{t \to t_0} |V(t) - V(t_0)| = 0$$

i.e., $\lim_{t \to 0} |V(t)| = V(t_0)$

We can write $V(t) = [V_1(t), V_2(t), V_3(t)]$

Here $V(t)$ is continuous at t_0 iff its three components at t_0.

Derivative :-

A vector function $V(t)$ is said to be differentiable at a point t if the following limit exist :

$$V'(t) = \lim_{\Delta t \to 0} \frac{V(t+\Delta t) - V(t)}{\Delta t}$$

$V'(t)$ is called the derivative of $V(t)$.

$$V'(t) = [V_1'(t), V_2'(t), V_3'(t)]$$

Curve :-

A curve C in space can be represented by a vector function,

$$r(t) = [x(t), y(t), z(t)]$$

$$\Rightarrow \boxed{r(t) = x(t)\, i + y(t)\, j + z(t)\, k}, \quad \text{—— (1)}$$

where x, y, z are cartesian co-ordinates.

This is called parametric representation of the curve and t is called the parameter of the representation.

To each value t_0 of t there corresponds a point of C with position vector $r(t_0)$, (i.e, with co-ordinates $x(t_0), y(t_0), z(t_0)$).

Also (1) gives an orientation of curve C, a direction of travelling along C so that t increases

This is called positive sense on C given by ① - That of decreasing t is the negative sense.

Example :-

A straight line L through a point A with position vector a in the direction of constant vector b can be represented in the form,

$$r(t) = a + tb$$
$$= [a_1 + tb_1, a_2 + tb_2, a_3 + tb_3]$$

If b is a unit vector, its components are the direction cosines of L.

Example :-

Find parametric representation of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0.$

Sol^n :- $x = a\cos\theta$, $y = b\sin\theta$, $z = 0$

$$r(t) = [a\cos\theta, b\sin\theta, 0]$$
$$= a\cos\theta i + b\sin\theta j$$

Example :-

Find parametric representation of a circle with center at $(2, 3)$ and radius 5.

Sol^n :- Eq^n of circle is,

$$(x-2)^2 + (y-3)^2 = 5^2$$

$x(t) = 2 + 5\cos\theta$ $\quad y(t) = 3 + 5\sin\theta$.

$r(t) = [2 + 5\cos\theta, 3 + 5\sin\theta] = (2 + 5\cos\theta)i + (3 + 5\sin\theta)j$

Tangent :-

The tangent vector of C,

$$\boxed{T = \gamma'(t)}$$

The unit tangent vector

$$\boxed{\hat{T} = \frac{\gamma'(t)}{|\gamma'(t)|}}$$

The tangent to C at P is given by,

$$\boxed{T(w) = \gamma + w\gamma'},$$

This is the sum of position vector γ of P and a multiple of tangent vector γ' of C at P. Both vectors depend on P. The variable w is the Parameter.

Example :-

Find tangent vector and unit tangent vector at given point P.

i) $\gamma(t) = ti + t^2 j$, $P(1,1,0)$

ii) $\gamma(t) = 2\cos t\, i + 2\sin t\, j$, $P(\sqrt{2}, \sqrt{2}, 0)$.

Solution :-

i) $\gamma(t) = ti + t^2 j$, $x(t) = t$, $y(t) = t^2$, $z(t) = 0$

$T = \gamma'(t) = i + 2tj$

Given $P(1,1,0)$

i.e., $x = 1$, $y = 1$, $z \cancel{(t)} = 0$

$\Rightarrow t = 1$ $\quad t^2 = 1$, $z = 0$

$\therefore \boxed{t = 1}$

~~$\therefore T(t) = r'(t) = i + 2j$~~

ii) ~~$r(t) =$~~ $\therefore \boxed{T = i + 2j}$

The unit tangent vector

$$\hat{T} = \frac{T}{|T|} = \frac{i + 2j}{\sqrt{1+4}} = \frac{1}{\sqrt{5}} i + \frac{2}{\sqrt{5}} j$$

ii) $r(t) = 2\cos t\, i + 2\sin t\, j$.

Here $x(t) = 2\cos t$, $y(t) = 2\sin t$, $z(t) = 0$

Given point is, $(\sqrt{2}, \sqrt{2}, 0)$

$\therefore x(t) = \sqrt{2}$, $y(t) = \sqrt{2}$, $z(t) = 0$

$\Rightarrow 2\cos t = \sqrt{2}$, $2\sin t = \sqrt{2}$, $z(t) = 0$

$\Rightarrow \cos t = \frac{1}{\sqrt{2}}$, $\sin t = \frac{1}{\sqrt{2}}$, $z(t) = 0$

$\Rightarrow t = \frac{\pi}{4}$.

Now Tangent vector,

$T = r'(t)$

$\Rightarrow T = -2\sin t\, i + 2\cos t\, j$

at $\pi/4$.

$T = -2 \cdot \frac{1}{\sqrt{2}} i + 2 \cdot \frac{1}{\sqrt{2}} j$

$\Rightarrow \boxed{T = -\sqrt{2} i + \sqrt{2} j}$

$|T| = \sqrt{2+2} = \sqrt{4} = 2$.

$\therefore \boxed{t=1}$

$\therefore$ ~~$T(t) = r'(t) = i + 2j$~~

ii) ~~$r(t) =$~~ $\therefore \boxed{T = i + 2j}$

The unit tangent vector

$$\hat{T} = \frac{T}{|T|} = \frac{i + 2j}{\sqrt{1+4}} = \frac{1}{\sqrt{5}} i + \frac{2}{\sqrt{5}} j$$

ii) $r(t) = 2\cos t\, i + 2\sin t\, j$.

Here $x(t) = 2\cos t$, $y(t) = 2\sin t$, $z(t) = 0$

Given point is, $(\sqrt{2}, \sqrt{2}, 0)$

$\therefore x(t) = \sqrt{2}$, $y(t) = \sqrt{2}$, $z(t) = 0$

$\Rightarrow 2\cos t = \sqrt{2}$, $2\sin t = \sqrt{2}$, $z(t) = 0$

$\Rightarrow \cos t = \frac{1}{\sqrt{2}}$, $\sin t = \frac{1}{\sqrt{2}}$, $z(t) = 0$

$\Rightarrow t = \frac{\pi}{4}$.

Now Tangent vector,

$T = r'(t)$

$\Rightarrow T = -2\sin t\, i + 2\cos t\, j$

at $\pi/4$.

$T = -2 \cdot \frac{1}{\sqrt{2}} i + 2 \cdot \frac{1}{\sqrt{2}} j$

$\Rightarrow \boxed{T = -\sqrt{2} i + \sqrt{2} j}$

$|T| = \sqrt{2+2} = \sqrt{4} = 2$.

The unit tangent vector,

$$\hat{T} = \frac{T}{|T|} = \frac{-\sqrt{2}\,i + \sqrt{2}\,j}{2}$$

$$\Rightarrow \boxed{\hat{T} = \frac{-1}{\sqrt{2}}\,i + \frac{1}{\sqrt{2}}\,j}$$

Example :-

Find tangent to the eliplae $\frac{x^2}{4} + y^2 = 1$ at $P : (\sqrt{2}, 1/\sqrt{2})$.

Soln :- The given equation of elipse is,

$$\frac{x^2}{4} + y^2 = 1.$$

The parametric representation of curve is,

$$r(t) = [2\cos t, \sin t]$$

The given point is, $P : (\sqrt{2}, \frac{1}{\sqrt{2}})$

$$\therefore x = \sqrt{2}, \quad y = \frac{1}{\sqrt{2}}$$

$$\Rightarrow 2\cos t = \sqrt{2}, \quad \sin t = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos t = \frac{1}{\sqrt{2}}, \quad \sin t = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \boxed{t = \frac{\pi}{4}}$$

$r'(t) = -2\sin t\, i + \cos t\, j$

The tangent to the elipse at P is,

$T(w) = r + wr'$

$= \left(2\cos\frac{\pi}{4}\, i + \sin\frac{\pi}{4}\, j\right) + w\left(-2\sin\frac{\pi}{4}\, i + \cos\frac{\pi}{4}\, j\right)$

$= \left(2\times\frac{1}{\sqrt{2}}\, i + \frac{1}{\sqrt{2}}\, j\right) + w\left(-2\times\frac{1}{\sqrt{2}}\, i + \frac{1}{\sqrt{2}}\, j\right)$

$= \sqrt{2}\, i + \frac{1}{\sqrt{2}}\, j - \sqrt{2}w\, i + \frac{w}{\sqrt{2}}\, j$

$$\boxed{T(w) = \sqrt{2}(1-w)\, i + \frac{1}{\sqrt{2}}(1+w)\, j}$$

Problem Set :-

1. Find parametric representation of the straight line through a point A in the direction of a vector b.

i) A : (4, 2, 0), b = i + j

ii) A: (-1, 3, 8), b = [3, 1, 0]

iii) A: (1, 1, 1), b = [-1, 1, 1]

iv) A : (1, 2, 3), b = 2i + j + k

2. Find parametric representation of straight line passing through A and B by using formula $r = A + (B-A)t$ and find the value of parameter t when it moves from A to B.

i) A : (2, 3, 0), B : (5, -1, 0)

ii) A: (1, 2, 3), B: (3, 2, 0)

3. For a given curve $C: r(t)$ find tangent vector, corresponding unit tangent vector and tangent vector to C at P.

i) $r(t) = t\,i + t^3 j$, $P: (1, 1)$

ii) $r(t) = 2\cos t\, i + 2\sin t\, j + tK$, $P: (2, 0, 0)$

iii) $r(t) = \cos t\, i + 2\sin t\, j$, $P: (\frac{1}{2}, \sqrt{3})$

iv) $r(t) = t\,i + t^2 j + t^3 K$, $P: (1, 1, 1)$

Length of a curve:

The length of a curve C is denoted by l and defined as

$$\boxed{l = \int_a^b \sqrt{r' \cdot r'}\; dt}$$

C is called rectifiable.

The arc length function is denoted by $S(t)$ and defined as,

$$\boxed{S(t) = \int_a^t \sqrt{r' \cdot r'}\; d\tilde{t}}$$

Example:

Find length of the given curve,

$r(t) = t\,i + \cosh t\, j$ from $t = 0$ to $t = 1$.

Sol^n:

$r(t) = t\,i + \cosh t\, j$

$r'(t) = i + \sinh t\, j$

$\gamma'(t) \cdot \gamma'(t) = 1 + \sinh^2 t$

$= \cosh^2 t$

$\therefore$ The length of the curve is,

$$l = \int_0^1 \sqrt{\cosh^2 t}\, dt$$

$$= \int_0^1 \cosh t \, dt$$

$$= \sinh t \Big]_0^1$$

$$= \sinh 1 - \sinh 0$$

$$= \sinh 1$$

Problems :-

Find lengths of the given curve,

1. $\gamma(t) = a\cos t\, i + a\sin t\, j + ct\, K$ from $(a, 0, 0)$ to $(a, 0, 2\pi c)$.

2. $\gamma(t) = t\, i + t^{3/2}\, j$ from $(0, 0, 0)$ to $(4, 8, 0)$.

3. $\gamma(t) = \cos t\, i + \sin t\, j$, from $(1, 0)$, to $(0, -1)$.

Gradient of a Scalar Field :-

The gradient of a scalar function function which is a vector is denoted by grad f / ∇f function and is defined as,

$$\boxed{\text{grad} f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k}.$$

Here we must assume that f is differentiable

The differential operator,

$$\boxed{\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k}$$

(read as nabla or delta)

$$\boxed{\text{grad} f = \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k}$$

Example:-

Find gradiant of $f = 2x + yz - 3y^2$

Soln:- $\frac{\partial f}{\partial x} = 2 \quad \frac{\partial f}{\partial y} = z - 6y \quad \frac{\partial f}{\partial z} = y$

$\therefore$ The gradiant of f is,

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$= 2i + (z-6y) j + y k.$$

Directional Derivative :-

The directional derivative of a scalar field f at P in the direction of the unit vector b is denoted by $D_b f$ and defined as,

$$\boxed{D_b f = \frac{df}{ds} = b \cdot \text{grad} f}$$

<u>Note</u> :- If the given direction vector is not a unit Vector. ~~then~~ then, convert it in to unit vector and calculate Directional derivative.

<u>Example</u> :-

Find Directional derivative of $z^2 = 4(x^2+y^2)$ at the Point $P:(1,0,2)$ in the direction of $\frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k$

<u>Sol^n</u> :- $z^2 = 4(x^2+y^2)$

$\Rightarrow 4(x^2+y^2) - z^2 = 0$

Here $f(x,y,z) = 4(x^2+y^2) - z^2$

$\frac{\partial f}{\partial x} = 8x \quad \frac{\partial f}{\partial y} = 8y \quad \frac{\partial f}{\partial z} = -2z$

$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k.$

$= 8x\, i + 8y\, j - 2z\, k.$

~~∇f at p = $8i - 2zk$~~

∇f at $p = 8i - 4k$

The given direction vector b is unit vector.

$\therefore D_b f = b \cdot \text{grad} f$

$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \cdot (8, 0, -4)$

$= \frac{8}{\sqrt{3}} + 0 - \frac{4}{\sqrt{3}} = \frac{4}{\sqrt{3}}.$

Example:-

Find directional derivative of $f = 2x^2 + 3y^2 + z^2$ at the point $P: (2, 1, 3)$ in the direction of the vector $a = i - 2k$,

Sol^n:-

$$\text{grad} f = 4xi + 6yj + 2zk$$

and at P,

$$\text{grad} f = 8i + 6j + 6k$$

Now, the direction vector is not an unit vector.

$$\therefore b = \frac{a}{|a|} = \frac{i - 2k}{\sqrt{5}} = \frac{1}{\sqrt{5}} i - \frac{2}{\sqrt{5}} k$$

$$\therefore D_b f = b \cdot \text{grad} f$$

$$= \left(\frac{1}{\sqrt{5}} i - \frac{2}{\sqrt{5}} k\right) \cdot (8i + 6j + 6k)$$

$$= \frac{8}{\sqrt{5}} - \frac{12}{\sqrt{5}} = \frac{-4}{\sqrt{5}}$$

Problem:-

Find gradient of f at P,

1. $f = x^2 - y^2$, $P: (-1, 3)$
2. $f = xy$, $P: (1, 1)$
3. $f = \ln(x^2 + y^2)$, $P: (2, 0)$
4. $f(x) = e^x \sin y$, $P: (\ln 2, \frac{\pi}{4})$

Find directional derivative of f at P in the direction of a.

1. $f = x^2 + y^2$, $P: (1, 1)$ $a = 2i - 4j$
2. $f = \ln(x^2 + y^2)$, $P: (4, 0)$, $a = i - j$

3. $f = x^2+3y^2+4z^2$, $P:(1,0,1)$, $a=-i-j+k$

4. $f = x-y$, $P:(4,5)$, $a = 2i+j$

Maximum and Minimum increase :-

Let $f(P) = f(x,y,z)$ be a scalar funtion having continuous first partial derivatives. Then grad f exists and is not the zero vector at a point. Then

i) At P, $f(x,y,z)$ has its Maximum rate of change in the direction of $\nabla f(P)$. The Maximum increase is $\|\nabla f(P)\|$.

ii) At P, $f(x,y,z)$ has its Minimum rate of change in the direction of $-\nabla f(P)$. This minimum rate of change is $-\|\nabla f(P)\|$

Example :-

Find Maximum and Minimum increase of $f = 2xz + e^y z^2$ from $(2,1,1)$.

Sol^n :- $\nabla f = 2zi + e^y z^2 j + (2x + 2ze^y)k$

$\nabla f(P) = 2i + ej + (4+2e)k$

The Maximum increase of f at $(2,1,1)$ is the direction of $\nabla f(P)$ and this Maximum increase is

$\sqrt{4+e^2+(4+2e)^2}$

The minimum rate of increase of f at $(2,1,1)$ is in the direction of $-\nabla f(P) = -2i - ej - (4+2e)k$ and this Minimum rate of change is $-\sqrt{4+e^2+(4+2e)^2}$.

Problem :-

Determine at this point Maximum and Minimum rate of change of the function.

1. $f = xyz$, $P:(1,1,1)$
2. $f = x^2y - \sin(xz)$, $P:(1,-1,\frac{\pi}{4})$
3. $f = 2xy + xe^z$, $(-2,1,6)$
4. $f = \cos(xyz)$, $(-1,1,\pi/2)$.

Gradient as Surface Normal Vector :-

Let f be a differentiable scalar function that represents a surface $S: f(x,y,z) = c$ = Constant. Then if the gradient of f at a point P of S is not the zero vector, it is a normal vector of S at P.

$$\therefore \boxed{\begin{aligned} n &= \text{grad} f \\ \hat{n} &= \text{Unit Normal vector} \\ &= \frac{n}{|n|} \end{aligned}}$$

Example :-

Find unit normal vector of $z^2 = 4(x^2+y^2)$ at the point $P:(1,0,2)$.

Solution :- $f(x,y,z) = 4(x^2+y^2) - z^2$.

$n = \nabla f = 8x\hat{i} + 8y\hat{j} - 2z\hat{k}$

at $P:(1,0,2)$

$\nabla f = 8\hat{i} - 4\hat{k}$

$\therefore$ The unit normal vector of f at p is,

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{8i - 4k}{\sqrt{64+16}} = \frac{8i-4k}{80} = \frac{8i-4k}{4\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}} i - \frac{1}{\sqrt{5}} k$$

Problems :-

Find normal vector and unit normal vector of the given curve,

1. $y = 1 - x^2$, P : (1, 0)
2. $z = x^2 - y^2$, P : (1, 1, 0)
3. $x^2 + y^2 + 2z^2 = 26$ P : (2, 2, 3)
4. $x^2 - y^2 + z^2 = 0$, P : (1, 1, 0)

Divergence of a Vector Field :-

Let V(x, y, z) be a differentiable vector function and let V_1, V_2, V_3 be components of V. The divergence of a vector field is a scalar function which is denoted by div V / $\nabla \cdot V$ and defined as,

$$\text{div} V = \nabla \cdot V$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot (V_1 i + V_2 j + V_3 k)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

$$\therefore \boxed{\text{div} V = \nabla \cdot V = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}}$$

Example :-

Find divergence of $x^2 i + y^2 j + z^2 k$

Sol^n:- Let $V = x^2 i + y^2 j + z^2 k$

$$\text{div } V = \nabla \cdot V$$
$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$
$$= 2x + 2y + 2z$$
$$= 2(x + y + z).$$

Problem :-

Find Divergence of

1. $xi + yj + zk$

2. $e^x \cos y \, i + e^x \sin y \, j + z k$

3. $e^x i + y e^{-x} j + 2z \sinh x \, k$

4. $xyz(xi + yj + zk)$

Note :-

→ $\text{div}(\text{grad} f) = \nabla^2 f$ [Laplacian of f]

→ The condition of Incompressibility is $\text{div}(V) = 0$.

→ Divergence Measures outflow minus inflow.

Curl of a Vector Field :-

Let V be a differentiable vector function Then curl of a vector function is denoted by curl V / $\nabla \times V$ which provides a vector function which is defined as,

$$\text{curl } v = \nabla \times v$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= i\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + j\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + k\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

Example:-

Find curl of $v = yz\,i + 3zx\,j + z\,k$.

Sol^n:-

$$\text{curl } v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(3zx)\right] + j\left[\frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(z)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(3zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$= i\,(0 - 3x) + j\,(y - 0) + k\,(3z - z)$$

$$= -3x\,i + y\,j + 2z\,k.$$

Theorem:-

Let f be continuous in its first and second partial derivatives then, $\text{curl}(\text{grad} f) = 0$.

Proof:- $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

L.H.S. $\text{Curl}(\text{grad} f) = \nabla \times \text{grad} f$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) + j\left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right) + k\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$$

$$= i(0) + j(0) + k(0)$$

$$= 0 = \text{R.H.S.} \text{ (proved)}$$

Th^m 1- Let V be a continuous vector field whose components have continuous first and second partial derivatives. Then $\operatorname{div}(\operatorname{curl} V) = 0$

Proof:- Let $V = [V_1, V_2, V_3]$ be a continuous differentiable function.

$$\operatorname{curl} V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= i\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + j\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + k\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

L.H.S. $\operatorname{div}(\operatorname{curl} V) = \nabla \cdot \operatorname{curl} V$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left[\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) k\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 v_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 v_2}{\partial x \partial z}} + \cancel{\frac{\partial^2 v_1}{\partial y \partial z}} - \cancel{\frac{\partial^2 v_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 v_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 v_1}{\partial z \partial y}}$$

$= 0 =$ R.H.S. (Proved)

Note :-

→ Condition of irrotation is $\text{curl}(V) = 0$

→ A field which has zero divergence everywhere is called solenoidal

Problems :-

Find curl of the given ~~curve~~ vector function.

1. $[2y, 5x, 0]$
2. $[\sin y, \cos z, 0]$
3. $xyz(xi + yj + zk)$
4. $[\sin x, xy + z, x^2 y^2]$