Fourier Analysis
Module - 5

Periodic Function :-

A function $f: R \to R$ is said to be periodic, if there exists some positive real number P such that $f(x+p) = f(x)$ for all real numbers x.

This number P is called a period of the periodic function $f(x)$.

The function, $f(x) = c$, where c is a constant is a periodic function as it satisfies for every positive P.

Familiar periodic functions are sine and cosine functions. For $\cos x$ and $\sin x$ the fundamental period is 2π.

Examples :- Sine, cosine, constant function, tangent are periodic.

exponential, non-constant polynomial are nonperiodic.

Note :-

- Linear combination of two periodic functions with same period is a periodic function of that period.
- If a periodic function $f(x)$ has a smallest period $P(>0)$ this is often called the fundamental period of $f(x)$.
- A function without fundamental period is $f(x) = c$.
- If p is period of $f(x)$, then np will be period of $f(x)$; where, $n = 1, 2, \ldots$

- If $f(x)$ and $g(x)$ are periodic functions of period P_1 and P_2 respectively and $h(x) = af(x) + bg(x)$, then $h(x)$ has period $= LCM(P_1, P_2)$.

Trigonometric Series :-

Let's consider various function of period $p = 2\pi$

$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots,$

By using the above functions, we can write a series as,

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots$$

$$= \boxed{a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx}, \quad \text{—— (1)}$$

where, $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are real constants.

Such a series is called a trignometric series and a_n and b_n are co-efficients of the series. Hence, if the series (1) has the period 2π. The series $f(x)$ converges, its sum will be a function of period 2π.

Results :-

$$* \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases}$$

$$* \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases}$$

$$* \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0$$

Fourier Series :

Fourier Series is a trigonometric series which represents a given periodic function f(x) in terms of cosine and sine functions.

Let's consider a periodic function f(x) defined on $-\pi \le x \le \pi$.

Let us further assume that f(x) can be represented by a trigonometric series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad -(1)$$

i.e., We assume that this series converges and f(x) has its sum.

We have to determine the co-efficients a_n and b_n of the corresponding series (1).

Integrating on both sides of (1) from $-\pi$ to π we will get,

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\right] dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\, dx = a_0\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\, dx + b_n \int_{-\pi}^{\pi} \sin nx\, dx$$

$$= 2\pi a_0 + \sum a_n \times 0 + b_n \times 0$$

$$= 2\pi a_0$$

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

Now multiply (1) by cosmx, where m is any fixed positive integer, and integrate from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx\right]\cos mx\,dx$$

$$= a_0\int_{-\pi}^{\pi}\cos mx + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx$$

$$= a_0\times 0 + a_1\times 0 + a_2\times 0 + \cdots + a_m\times\pi + \cdots$$

$$+ b_1\times 0 + b_2\times 0 + \cdots +$$

$$= a_m\pi$$

$$\Rightarrow \boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx}$$

Now multiply (1) by Sinmx, where m is any fixed positive integer and integrate from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx\right]\sin mx\,dx$$

$$= a_0\int_{-\pi}^{\pi}\sin mx\,dx + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nx\sin mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\sin mx\,dx$$

$$= b_m\pi$$

$$\Rightarrow \boxed{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx}$$

Now, the trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx,$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

is called as Fourier series of $f(x)$ and a_0, a_n, b_n are Fourier co-efficients of $f(x)$.

Example 1:- Find fourier series expansion of

$f(x) = x^2$, $(-\pi, \pi)$ of period 2π.

Solution :-

Fourier series of $f(x)$ in $(-\pi, \pi)$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} 2x\,\frac{\sin nx}{n}\,dx$$

$$= -\frac{2}{n\pi}\int_{-\pi}^{\pi} x\sin nx\,dx$$

$$= -\frac{2}{n\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= -\frac{2}{n\pi}\left[-\left\{\frac{\pi\cos n\pi}{n} + \pi\frac{\cos n\pi}{n}\right\} + 0\right]$$

$$= \frac{4}{n^2}\cos n\pi$$

$$= \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx = 0$$

$$\cancel{= \frac{1}{\pi}\left[\left\{-x^2\frac{\cos nx}{n} - 2\right\}\right.}$$

$\therefore$ The required Fourier series is,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

Example :

Find Fourier series of

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Sol^n :-

The fourier series of $f(x)$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} -\pi\, dx + \int_{0}^{\pi} x\, dx\right]$$

$$= \frac{1}{2\pi}\left[-\pi\int_{-\pi}^{0} 1\, dx + \int_{0}^{\pi} x\, dx\right]$$

$$= \frac{1}{2\pi}\left[-\pi \times (\pi) + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{2\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right)$$

$$= \frac{1}{2\pi} \times \left(\frac{-\pi^2}{2}\right) = \frac{-\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\, dx + \int_{0}^{\pi} f(x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\cos nx\, dx + \int_{0}^{\pi} x \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\frac{\sin n x}{n}\Big|_{-\pi}^{0} + \left\{ x\frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right\}_0^{\pi} \right]$$

$$= \frac{1}{\pi}\left[0 + 0 + \frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right]$$

$$= \frac{1}{n^2\pi}[\cos n\pi - 1]$$

$$= \frac{1}{n^2\pi}[(-1)^n - 1]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\, dx + \int_0^{\pi} f(x)\sin nx\, dx \right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\sin nx\, dx + \int_0^{\pi} x\sin nx\, dx \right]$$

$$= \frac{1}{\pi}\left[-\pi\cdot\left(\frac{-\cos nx}{n}\right)\Big|_{-\pi}^{0} + \left\{ x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2} \right\}_0^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) + \frac{1}{n}(-\pi\cos n\pi - 0) + \frac{1}{n^2}(\sin\pi - 0) \right]$$

$[\cos(-n\pi) = \cos n\pi,\ \sin n\pi = 0]$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - (-1)^n) - \frac{\pi}{n}(-1)^n \right]$$

$$= \frac{1}{n}[1 - 2(-1)^n]$$

$\therefore$ The Fourier series of $f(x)$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$$

$$\Rightarrow f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{\{(-1)^n-1\}}{n^2\pi}\cos nx + \frac{1}{n}(1-2(-1)^n)\sin nx\right]$$

~~But~~ At $x=0$, the point of discontinuity,

$$f(0) = \frac{f(0^+)+f(0^-)}{2}$$

$f(0^+)$ = Left Hand Limit of $f(x)$ at $x=0$

$$= \frac{-\pi+0}{2}$$

$f(0^-)$ = Right Hand Limit of $f(x)$ at $x=0$

$$= -\frac{\pi}{2}$$

Orthogonality of the Trigonometric System :-

The trigonometric system

$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,$ ———

is orthogonal on $-\pi \le x \le \pi$.

i.e., $\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = 0 \quad (m \ne n)$

$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0 \quad (m \ne n)$

and $\int_{-\pi}^{\pi} \cos mx \sin nx\,dx = 0$, for any integer m and n.

Convergence of Fourier Series :-

If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \le x \le \pi$ and has a left hand derivative and right hand derivative at each point of that interval then fourier series of $f(x)$ is convergent. Its sum is $f(x)$, except at a point x_0 at which $f(x)$

is discontinuous and the sum of the series is the average of left hand limit and right hand limit of $f(x)$ at $x = x_0$.

Condition for Fourier Series :-

A function $f(x)$ can be expanded in Fourier series, $a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$,

where a_0, a_n, b_n are constants,

if provided that,

1. $f(x)$ is periodic, single valued and finite.
2. $f(x)$ has finite number of discontinuity
3. $f(x)$ has the atmost a finite number of Maxima and Minima.

These conditions are known as ~~Bars~~ Dirichlet's Condition for Fourier series of a function $f(x)$.

Fourier Series for Functions of any period $P = 2L$

For periodic function $f(x)$ of period $2L$, the Fourier Series is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x$$

with Fourier Co-efficients of $f(x)$ given by,

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

Proof :- Let's Consider a periodic function $g(v)$ of period $P = 2\pi$. Then Fourier series of $g(v)$ is,

$$f(v) = a_0 + \sum_{n=1}^{\infty} a_n \cos nv + b_n \sin nv$$

where
$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v)\, dv$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \cos nv\, dv$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \sin nv\, dv.$$

Let $v = \frac{\pi x}{L} \Rightarrow x = \frac{Lv}{\pi}$

when $v = \pi$, $x = L$ and $v = -\pi$, $x = -L$.

Here $x = \pm L$ corresponds to $v = \pm\pi$.

Thus f, regarded as a function of v that we call $g(v)$,

$$f(x) = g(v).$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)\, dv$$

$$= \frac{1}{2\pi}\int_{-L}^{L} f(x)\, \frac{\pi}{L}\, dx$$

$$\Rightarrow \boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v) \cos nv\, dv$$

$$= \frac{1}{\pi}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, \frac{\pi}{L}\, dx$$

$$\Rightarrow \boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v) . \sin nv\, dv$$

$$= \frac{1}{\pi}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, \frac{\pi}{L}\, dx$$

$$\Rightarrow \boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx}$$

Note :-

Interval of Integration :- We may replace the ~~interval~~ interval of integration by any interval of length $P = 2L$. For example by interval $0 \le x \le 2L$.

Even function :

→ A function $f(x)$ is even iff $f(-x) = f(x)$ for all x in the domain of f.

→ Even function is ~~even~~ symmetric with respect to the y-axis.

→ Ex :- $\cos x$, x^2, x^4, $|x|$, x^2+1

→ A function f is odd iff the graphs of $f(x)$ and $-f(-x)$ coincide.

→ Integration ~~of~~ Even function.

f is ~~always~~ continuous on $[-a, a]$

then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

ODD Function :

→ A function $f(x)$ is odd iff $f(-x) = -f(x)$ for all x in the domain of f.

→ Odd function is symmetric with respect to origin.

→ Ex :- $\sin x$, x^3, x^5, --- $x^3 + x$.

→ A function f is even iff the graphs of $f(x)$ and $f(-x)$ coincide.

→ Integration of odd function

f is continuous on $[-a, a]$

then,

$$\int_{-a}^{a} f(x)\,dx = 0$$

Example :

Find Fourier series of the function

$$f(x) = \begin{cases} 0, & \text{if } -2 < x < -1 \\ K, & \text{if } -1 < x < 1 \\ 0, & \text{if } 1 < x < 2, \end{cases}$$

$P = 2L = 4, \quad L = 2.$

Solution :

The fourier series of given $f(x)$ can be written as,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2}.$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} K\,dx$$

$$= \frac{K}{4} \times 2 = \frac{K}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} K\cos\frac{n\pi x}{2}\,dx = \frac{K}{2}\int_{-1}^{1}\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{K}{2}\left[\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_{-1}^{1} = \frac{K}{n\pi}\left[\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right]$$

$$= \frac{2K}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} K\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{K}{2}\int_{-1}^{1} \sin\frac{n\pi x}{2}\,dx$$

$$= 0$$

Note :-

✓ product of even and odd function is odd

Fourier Cosine Series :

If a given periodic function of period $p = 2L$ is even, then the Fourier series will be,

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}}, \quad \text{——— (A)}$$

With coefficients,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\,2\int_{0}^{L} f(x)\,dx$$

$$\Rightarrow \boxed{a_0 = \frac{1}{L}\int_{0}^{L} f(x)\,dx}$$

L

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$\Rightarrow \boxed{a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx}$$

Note:- If a periodic function is even then, $b_n = 0$

Now, Fourier Cosine Series for Even periodic function $f(x)$ with period $2L$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

where $a_0 = \frac{1}{L}\int_0^L f(x)\,dx$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

Fourier Sine Series:-

If a given periodic function with period $p=2L$ is odd then a_0 and a_n will be 0 and the Fourier Series will be,

$\cancel{f(x) = a_0 + \sum}$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

which is called as Fourier Sine Series.

Sum of functions:-

The Fourier coefficients of a sum of $f_1 + f_2$ are the sums of the corresponding Fourier Co-efficients of f_1 and f_2.

The Fourier Coefficients of cf are c times the corresponding Fourier Co-efficients of f.

Example :-

Find Fourier series of $f(x) = x, \quad -\pi < x < \pi$

and $f(x+2\pi) = f(x)$

Solution :-

Here the given function is odd.

Therefore the Fourier series of $f(x) = x$ will be Fourier sine series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cancel{dx}$$

$$= \sum_{n=1}^{\infty} b_n \sin nx$$

where, $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi} \left[x \cdot \left(-\frac{\cos nx}{n} \right) + \frac{\sin nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\frac{\pi}{n} \cos n\pi + 0 + \frac{\sin n\pi}{n^2} - 0 \right]$$

$$= -\frac{2}{n} \cos n\pi$$

$$= -\frac{2}{n} (-1)^n$$

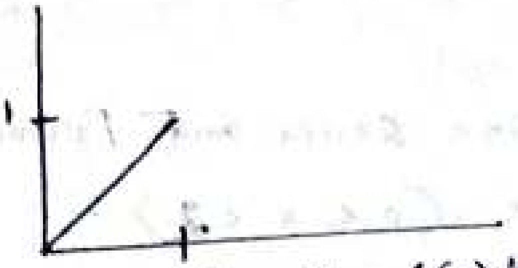
$$= \frac{2}{n} (-1)^{n+1}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx$$

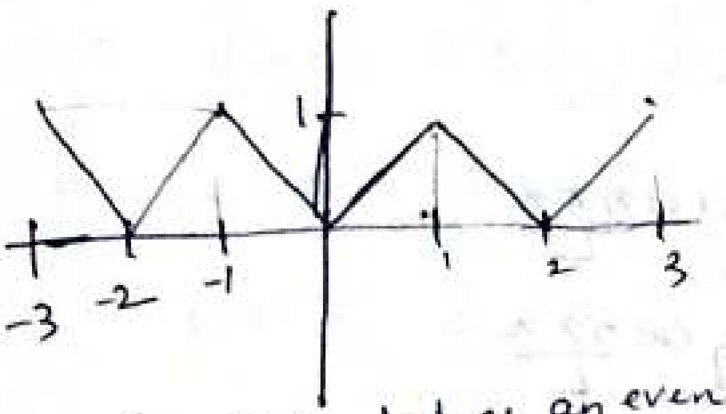
Half - Range Expansions :

Let us consider a function $f(x)$ defined in $0 \le x \le L$

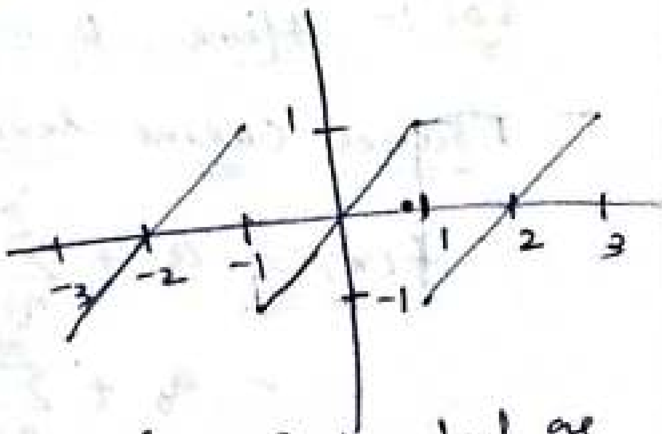
Let $f(x) = x, \; 0 \le x \le 1$



Given function $f(x)$



$f(x)$ Extended as an even Periodic function of period 2.



$f(x)$ Extended as an odd periodic function of period 2.

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ -x, & -1 \le x \le 0 \end{cases}$$

$p = 2 \quad f(x+2) = f(x)$

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ x, & -1 \le x < 0 \\ -x, & -1 \le x \le 0 \end{cases}$$

Note :

→ Even extension to the full range $-L \le x \le L$ from half range $0 \le x \le L$ and the periodic extension of the period $2L$ to the x-axis gives Fourier Cosine series.

→ Odd extension to $-L \le x \le L$ from $0 \le x \le L$ and the periodic extension of period $2L$ to the x-axis gives Fourier Sine Series.

Example :-

Find Fourier Cosine Series and Fourier Sine series of $f(x) = x$, $(0 < x < 2)$.

Sol^n :- Here $L = 2$

Fourier Cosine Series :-

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\, dx$$

$$= \frac{1}{2}\int_0^2 f(x)\, dx$$

$$= \frac{1}{2}\int_0^2 x\, dx = \frac{1}{2}\,\frac{4}{2} = 1$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\, dx$$

$$= \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\, dx$$

$$= \int_0^2 f(x)\cos\frac{n\pi x}{2}\, dx$$

$$= \int_0^2 x \cos\frac{n\pi x}{2}\, dx$$

$$= x \sin\frac{n\pi x}{2} \cdot \frac{2}{n\pi}\Big|_0^2 + \frac{4}{n^2\pi^2} \cos\frac{n\pi x}{2}\Big|_0^2$$

$$= \frac{4}{n\pi} \sin n\pi - 0 + \frac{4}{n^2\pi^2}(\cos n\pi - 1)$$

$$= \frac{4}{n^2\pi^2}((-1)^n - 1)$$

$\therefore$ Fourier cosine series for given function is,

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4[(-1)^n - 1]}{n^2\pi^2} \cos\frac{n\pi x}{2}$$

Fourier Sine Series :-

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

$$= \int_0^2 f(x) \sin\frac{n\pi x}{2}\, dx = \int_0^2 x \sin\frac{n\pi x}{2}\, dx$$

$$= x \cdot -\cos\frac{n\pi x}{2} \cdot \frac{2}{n\pi}\Big|_0^2 + \frac{4}{n^2\pi^2} \sin\frac{n\pi x}{2}\Big|_0^2$$

$$= -\frac{2}{n\pi}[2\cos n\pi - 0] + \frac{4}{n^2\pi^2}(\sin n\pi - 0)$$

$$= -\frac{4}{n\pi} \cos n\pi = \frac{4}{n\pi}(-1)^{n+1}$$

$\therefore$ Fourier Sine series for given function is,

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{2}$$

Assignment 6

1. Find smallest period of

i) $\cos 5x$ ii) $\sin \pi x$ iii) $\cos \frac{2\pi x}{k}$

iv) 5 v) $\sin x + \cos x$

vi) $\cos 2x + \sin x$

2. Find fourier series of given periodic function of period 2π.

i) $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 < x < \pi \end{cases}$

ii) $f(x) = \begin{cases} -1, & 0 < x < \pi/2 \\ 0, & \frac{\pi}{2} < x < 2\pi \end{cases}$

iii) $f(x) = \begin{cases} x^2, & \text{if } 0 < x < \pi \\ 4, & \text{if } \pi < x < 2\pi \end{cases}$

iv) $f(x) = x^3, \ -\pi < x < \pi$

3. Find Fourier series for period $p = 2L$

i) $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$ $p = 2$

ii) $f(x) = |x|, \ -2 < x < 2, \ p = 4$

iii) $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1 - x, & 1 < x < 2 \end{cases}$, $p = 2$

iv) $f(x) = 3x^2, \ -1 < x < 1, \ p = 2$

v) $f(x) = 1 - x^2, \ -1 < x < 1, \ p = 2$

vi) $f(x) = \pi \sin \pi x, \ 0 < x < 1, \ p = 1$

4. check the given functions are even, odd or neither even nor odd :

i) x^2 ii) $x^2 + \cos 2x$

iii) $x + x^3$ iv) e^x v) $x \sin x$

vi) e^{x^2} vii) x^2; $0 < x < 2\pi$

5. State whether the given function is even or odd. Find its Fourier series.

i) $f(x) = \begin{cases} K, & \text{if } -\pi \leq x \leq 0 \\ 0, & \text{if } 0 < x < \pi \end{cases}$

ii) $f(x) = \begin{cases} -2x, & -\pi < x < 0 \\ 2x, & 0 < x < \pi \end{cases}$

iii) $f(x) = \frac{x^2}{2}$, $-\pi < x < \pi$

iv) $f(x) = x + \sin x$, $-\pi < x < \pi$

6. Find the Fourier cosine series as well as sine series. Sketch $f(x)$ and its two periodic extensions.

i) $f(x) = x$, $0 < x < L$ ii) $f(x) = x^2$, $0 < x < L$

iii) $f(x) = \pi - x$, $0 < x < L$ iv) $f(x) = e^x$, $0 < x < L$.

Parseval's Identity for Fourier series :-

The fourier series of $f(x)$ in a interval $-L<x<L$ is defined by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\,dx$$

If fourier series of $f(x)$ converge uniformly in $(-L, L)$ then

$$\int_{-L}^{L} [f(x)]^2\,dx = L\left[2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2\right]$$

Proof :- $[f(x)]^2 = a_0 f(x) + \sum_{n=1}^{\infty} a_n f(x) \cos\frac{n\pi x}{L} + b_n f(x) \sin\frac{n\pi x}{L}$

$$\Rightarrow \int_{-L}^{L} [f(x)]^2\,dx = a_0 \int_{-L}^{L} f(x)\,dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\,dx + \sum_{n=1}^{\infty} b_n \int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\,dx$$

$$= a_0\, 2La_0 + \sum_{n=1}^{\infty} a_n \cdot La_n + \sum_{n=1}^{\infty} b_n \cdot Lb_n$$

$$= L\left[2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$$

$$\Rightarrow \int_{-L}^{L} [f(x)]^2\,dx = L\left[2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right],$$

(Parseval's Identity)

In Interval $(-l, l)$

$$\int_{-l}^{l}[f(x)]^2 dx = L\left[2a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right]$$

Half Range cosine series in $(0, l)$:-

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Parseval Identity :-

$$\int_0^L [f(x)]^2 dx = \frac{L}{2}\left[2a_0^2 + \sum_{n=1}^{\infty} a_n^2\right]$$

where, $a_0 = \frac{1}{L}\int_0^L f(x)\,dx$, $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{L}\,dx$

Half Range Sine series in $(0, l)$:-

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Parseval Identity :-

$$\int_0^L [f(x)]^2 dx = \frac{L}{2}\sum_{n=1}^{\infty} b_n^2 ,$$

where, $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$.

Example :-

For $f(x) = x^2$, $-\pi < x < \pi$, show that

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$$

Solution :-

Fourier coefficients of $f(x) = x^2$, $-\pi < x < \pi$ ~~are~~

are, $a_0 = \frac{\pi^2}{3}$ $a_n = \frac{4(-1)^n}{n^2}$, $b_n = 0$.

By Parseval Identity for $f(x)$ in $(-\pi, \pi)$

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \pi \left[2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \right]$$

$$= \pi \left[\frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{4^2}{n^4} \right]$$

$$\Rightarrow \int_{-\pi}^{\pi} x^4 \, dx = \frac{2\pi^5}{9} + \pi \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\Rightarrow \frac{x^5}{5} \Big|_{-\pi}^{\pi} = \frac{2}{9}\pi^5 + \pi \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\Rightarrow \frac{2\pi^5}{5} - \frac{2\pi^5}{9} = 16\pi \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \frac{8\pi^5}{45} = 16\pi \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\Rightarrow \frac{1}{1^4} + \frac{1}{2^4} + \cdots = \frac{\pi^4}{90}$$

Fourier Integrals :

Fourier series are powerful tools in treating various problems involving periodic functions. There are many problems which involve nonperiodic functions. Fourier integral is the extension of Fourier series of $f(x)$ with period $2L$, when $L \to \infty$.

From Fourier Series to Fourier Integral :

Now consider any periodic function $f_L(x)$ with period $2L$ which can be represented by a Fourier series,

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos w_n x + b_n \sin w_n x,$$

$$\text{where } w_n = \frac{n\pi}{L}$$

Now put the values of a_n and b_n and denote the variable of integration by v in the above Fourier series of $f_L(x)$,

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\Big[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v\,dv$$

$$+ \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v\,dv\Big]$$

Now let,

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L}$$

$$\Rightarrow \boxed{\Delta w = \frac{\pi}{L}}$$

$$\Rightarrow \frac{1}{L} = \frac{\Delta w}{\pi}$$

$$\therefore f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[(\cos w_n x)\,\Delta w \int_{-L}^{L} f_L(v)\cos w_n v\,dv$$

$$+ (\sin w_n x)\,\Delta w \int_{-L}^{L} f_L(v)\sin w_n v\,dv\Big] \quad (1)$$

This representation is valid for any fixed L, arbitrarily large, but finite.

Now let $L \to \infty$ and assume that the resulting periodic function

$$f(x) = \lim_{L\to\infty} f_L(x)$$

is absolutely integrable on the x-axis i.e., the following limit exist:

$$\int_{-\infty}^{\infty} |f(x)|\,dx = \lim_{a\to-\infty}\int_{a}^{0} |f(x)|\,dx + \lim_{b\to\infty}\int_{0}^{b} |f(x)|\,dx \quad (2)$$

Then $\frac{1}{L} \to 0$ and the value of the first Integral on the right hand side of (1) approches zero.

$$\left(\frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv \longrightarrow 0\right)$$

Also $\Delta w = \frac{\pi}{L} \to 0$ and infinite series in (1) becomes an integral from 0 to ∞ which represents $f(x)$, namely,

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv\,dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv\,dv\right]dw$$

Fourier Integral of $f(x)$ is,

$$\boxed{f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega, \quad \text{where, } A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv \quad \text{and } B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv} \quad (3)$$

Sufficient Condition for Existence of Fourier Integral:

If $f(x)$ is piecewise continuous in every finite interval and has a right-hand derivative and left-hand derivative at every point and if the integral (2) exists, then $f(x)$ can be represented by a Fourier Integral (3). At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the Left and right hand limits of $f(x)$ at that point.

Example:-

Find Fourier Integral Representation of

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Solution:-

The Fourier Integral of $f(x)$ is given by,

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega,$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv$$

$$= \frac{1}{\pi}\int_{-1}^{1} \cos wv \, dv$$

$$= \frac{\sin wv}{\pi w}\Big|_{-1}^{1} = \frac{2\sin w}{\pi w}$$

$$B(w) = \frac{1}{\pi}\int_{-1}^{1} \sin wv \, dv = 0$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos wx \sin w}{w} \, dw$$

At the point of discontinuity $x = 1$,

$$f(1) = \frac{f(1^+) + f(1^-)}{2}$$

$$= \frac{0 + 1}{2}$$

$$= \frac{1}{2}.$$

$$\therefore \quad \frac{2}{\pi}\int_0^\infty \frac{\cos wx \sin w}{w} \, dw = \begin{cases} 1, & 0 \le |x| < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & |x| > 1 \end{cases}$$

$$\Rightarrow \int_0^\infty \frac{\cos wx \sin w}{w} \, dw = \begin{cases} \pi/2, & 0 \le |x| < 1 \\ \pi/4, & x = 1 \\ 0, & |x| > 1. \end{cases}$$

is the required Fourier Integral Representation.

Note :-

- $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (b \sin bx + a \cos bx) + c$

- $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) + c$

- $f(a^+) \to$ Right Hand limit of $f(x)$ at $x=a$
- $f(a^-) \to$ Left Hand limit of $f(x)$ at $x=a$.

Example :-

Find Fourier Integral representation of

$$f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$$

Sol^n :- The fourier Integral representation of $f(x)$ is,

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x] \, d\omega,$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v \, dv$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} f(v)\cos\omega v \, dv + \int_0^\infty f(v)\cos\omega v \, dv\right]$$

$$= \frac{1}{\pi}\int_0^\infty e^{-v}\cos\omega v . dv$$

$$= \frac{1}{\pi}\left[\frac{e^{-v}}{1+\omega^2}(\omega\sin\omega v - \cos\omega v)\right]_0^\infty$$

$$= \frac{1}{\pi(1+\omega^2)}\left[e^{-v}(\omega\sin\omega v - \cos\omega v)\right]_0^\infty$$

$$= \frac{1}{\pi(1+\omega^2)}\left[0 . - 1(0-1)\right] = \frac{1}{\pi(1+\omega^2)}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v \, dv$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} f(v)\sin\omega v \, dv + \int_{0}^{\infty} f(v)\sin\omega v \, dv\right]$$

$$= \frac{1}{\pi}\int_{0}^{\infty} f(v)\sin\omega v \, dv$$

$$= \frac{1}{\pi}\int_{0}^{\infty} e^{-v}\sin\omega v \, dv$$

$$= \frac{1}{\pi}\left[\frac{e^{-v}}{1+\omega^2}(-\sin\omega v - \omega\cos\omega v)\right]_0^{\infty}$$

$$= \frac{1}{\pi(1+\omega^2)}\left[e^{-v}(-\sin\omega v - \omega\cos\omega v)\right]_0^{\infty}$$

$$= \frac{1}{\pi(1+\omega^2)}\left[0 - 1(0 - \omega)\right]$$

$$= \frac{\omega}{\pi(1+\omega^2)}.$$

$\therefore$ The Fourier Integral of $f(x)$ is,

$$f(x) = \int_{0}^{\infty}\left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right] d\omega$$

$$= \int_{0}^{\infty}\left[\frac{1}{\pi(1+\omega^2)}\cos\omega x + \frac{\omega}{\pi(1+\omega^2)}\sin\omega x\right] d\omega$$

$$= \int_{0}^{\infty}\frac{1}{\pi(1+\omega^2)}(\cos\omega x + \omega\sin\omega x) \, d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} \, d\omega$$

Now at the point of discontinuity $x=0$,

$$f(0) = \frac{f(0^-) + f(0^+)}{2}$$

$$= \frac{0+1}{2} = \frac{1}{2}.$$

$\therefore$ the Fourier Integral Representation of $f(x)$ is,

$$\frac{1}{\pi}\int_0^\infty \frac{\cos wx + w\sin wx}{1+w^2}\,dw = \begin{cases} 0, & x<0 \\ \frac{1}{2}, & x=0 \\ e^{-x}, & x>0 \end{cases}$$

$$\Rightarrow \int_0^\infty \frac{\cos wx + w\sin wx}{1+w^2}\,dw = \begin{cases} 0, & x<0 \\ \pi/2, & x=0 \\ \pi e^{-x}, & x>0 \end{cases}$$

Fourier Cosine Integral :-

If $f(x)$ is an even function, then $B(w)=0$.
Then the Fourier Integral reduced to

$$\boxed{\begin{aligned} f(x) &= \int_0^\infty A(w)\cos wx\,dw, \\ \text{where, } A(w) &= \frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv \end{aligned}}$$

is known as Fourier Cosine Integral.

Fourier Sine Integral :-

If $f(x)$ is an odd function, then $A(w)=0$

then the Fourier Integral reduced to,

$$\boxed{f(x) = \int_0^\infty B(\omega)\sin\omega x \, d\omega; \quad \text{where } B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v \, dv}$$

is Known as Fourier Sine Integral.

Example :-

Find the Fourier Cosine and Sine Integral represention of $f(x) = e^{-Kx}$, $x>0$, $K>0$.

Solution :-

Here $f(x) = e^{-Kx}$, $(K>0, x>0)$

Fourier Cosine Integral Representation :-

The Fourier Cosine Integral of $f(x)$ is,

$$f(x) = \int_0^\infty A(\omega)\cos\omega x \, d\omega,$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos\omega v \, dv$$

$$= \frac{2}{\pi}\int_0^\infty e^{-Kv}\cos\omega v \, dv$$

$$= \frac{2}{\pi}\left[\frac{e^{-Kv}}{K^2+\omega^2}(\omega\cos\omega v - K\cos\omega v)\right]_0^\infty$$

$$= \frac{2}{\pi(K^2+\omega^2)}\left[e^{-Kv}(\omega\sin\omega v - K\cos\omega v)\right]_0^\infty$$

$$= \frac{2}{\pi(K^2+\omega^2)}\left[0 - 1(0 - K)\right]$$

$$= \frac{2K}{\pi(K^2+\omega^2)}$$

$\therefore$ The Fourier Cosine Integral representation of $f(x)$ is,

$$f(x) = \int_0^\infty \frac{2K}{\pi(K^2+\omega^2)} \cos\omega x \, d\omega$$

$$\Rightarrow e^{-Kx} = \frac{2K}{\pi}\int_0^\infty \frac{\cos\omega x}{K^2+\omega^2}\, d\omega \quad [K>0,\ x>0]$$

$$\Rightarrow \int_0^\infty \frac{\cos\omega x}{K^2+\omega^2}\, d\omega = \frac{\pi}{2K} e^{-Kx} \quad [K>0,\ x>0].$$

Fourier Sine Integral Representation :-

The fourier sine integral of $f(x)$ is,

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\, d\omega,$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\, dv$$

$$= \frac{2}{\pi}\int_0^\infty e^{-Kv}\sin\omega v\, dv$$

$$= \frac{2}{\pi}\left[\frac{e^{-Kv}}{K^2+\omega^2}(-K\sin\omega v - \omega\cos\omega v)\right]_0^\infty$$

$$= \frac{2}{\pi(K^2+\omega^2)}\left[e^{-Kv}(-K\sin\omega v - \omega\cos\omega v)\right]_0^\infty$$

$$= \frac{2}{\pi(K^2+\omega^2)}[0 - 1(-\omega)]$$

$$= \frac{2\omega}{\pi(K^2+\omega^2)}$$

$$\therefore \boxed{B(\omega) = \frac{2\omega}{\pi(K^2+\omega^2)}}$$

$\therefore$ The Fourier Integral representation of $f(x)$ is,

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\, d\omega = \int_0^\infty \frac{2\omega}{\pi(k^2+\omega^2)} \sin\omega x\, d\omega$$

$$\Rightarrow \cancel{e^{-kx} = \frac{2}{\pi(k^2+\omega^2)}\int_0^\infty \frac{\sin\omega x}{k^2+\omega^2}\, d\omega}$$

$$\Rightarrow e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{\omega \sin\omega x}{k^2+\omega^2}\, d\omega \qquad (x>0, k>0)$$

$$\Rightarrow \int_0^\infty \frac{\omega\sin\omega x}{k^2+\omega^2}\, d\omega = \frac{\pi}{2} e^{-kx} \quad (x>0, k>0$$

Problem Set :-

$\rightarrow$ Show that the given integrals represent the indicated functions.

1. $$\int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}\, d\omega = \begin{cases} 0, & x<0 \\ \pi/2, & x=0 \\ \pi e^{-x}, & x>0 \end{cases}$$

2. $$\int_0^\infty \frac{\sin\omega \cos\omega x}{\omega}\, d\omega = \begin{cases} \pi/2, & 0\le x<1 \\ \pi/4, & x=1 \\ 0, & x>1 \end{cases}$$

3. $$\int_0^\infty \frac{(1-\cos\pi\omega)}{\omega}\sin\omega x\, d\omega = \begin{cases} \pi/2, & 0<x<\pi \\ 0, & x>\pi \end{cases}$$

4. $$\int_0^\infty \frac{\cos x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x} \text{ if } x>0.$$

$\rightarrow$ Find Fourier cosine Integral Representation of,

5. $$f(x) = \begin{cases} 1, & 0<x<1 \\ 0, & x>1 \end{cases}$$

6. $f(x) = \begin{cases} x^2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

7. $f(x) = e^{-x} + e^{-2x}, \quad (x > 0)$

→ Find Fourier Sine Integral Representation of

8. $f(x) = \begin{cases} x, & 0 < x < a \\ 0, & x > a \end{cases}$

9. $f(x) = \begin{cases} e^{-x}, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

10. $f(x) = \begin{cases} e^{x}, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

11. $f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$

12. $f(x) = \begin{cases} \pi - x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$

Fourier Cosine and Sine Transform :

Fourier Cosine and Sine Transform are the integral transform.

Fourier cosine Transform :

$$f(x) = \int_0^\infty A(w) \cos wx \, dw,$$

$$= \int_0^\infty \left[\frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv \right] \cos wx \, dw$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(v) \cos wv \, dv \right] \cos wx \, dw$$

Let us denote, $$\boxed{\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(v) \cos wv \, dv} \quad \text{--- (1)}$$

Now, $$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw} \quad \text{--- (2)}$$

Here, $\hat{f}_c(w)$ is called the Fourier Cosine transform of $f(x)$. Formula (2) gives back $f(x)$ from $\hat{f}_c(w)$. Therefore $f(x)$ is called the Inverse Fourier Cosine transform of $\hat{f}_c(w)$.

The process of obtaining the transform $\hat{f}_c$ (with variable w) from a given function f (with variable x) is also called Fourier cosine transformation.

Fourier Sine Transform :-

$$f(x) = \int_0^\infty B(w) \sin wx \, dw$$

$$= \int_0^\infty \left[\frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv\right] \sin wx \, dw$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(v) \sin wv \, dv\right] \sin wx \, dw$$

Let us denote, $\boxed{\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(v) \sin wv \, dv}$ — (3)

Now, $\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw}$ — (4)

Here, $\hat{f}_s(w)$ is called the Fourier Sine transform of $f(x)$. Formula (4) gives back ~~f(x)~~ from $\hat{f}_s(w)$. Therefore $f(x)$ is called the Inverse Fourier Sine Transform of $\hat{f}_s(w)$.

The process of obtaining the transform $\hat{f}_s$ (with variable w) from a given function f (with variable x) is called Fourier Sine Transformation.

Example :-

Find Fourier Cosine and Fourier Sine transform of $f(x) = \begin{cases} K, & \text{if } 0 < x < 2 \\ 0, & \text{if } x > 2 \end{cases}$

Solution :-

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^2 f(x) \cos wx \, dx + \int_2^\infty f(x) \cos wx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^2 K \cos wx \, dx$$

$$= K\sqrt{\frac{2}{\pi}} \int_0^2 \cos wx \, dx$$

$$= K\sqrt{\frac{2}{\pi}} \left[\frac{\sin wx}{w} \right]_0^2$$

$$= \frac{K}{w}\sqrt{\frac{2}{\pi}} \left[\sin 2w - 0 \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{K \sin 2w}{w}$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^2 K \sin wx \, dx$$

$$= K\sqrt{\frac{2}{\pi}} \int_0^2 \sin wx \, dx$$

$$= K\sqrt{\frac{2}{\pi}} \left[-\frac{\cos wx}{w} \right]_0^2$$

$$= \frac{K}{w}\sqrt{\frac{2}{\pi}} \left[-\cos 2w + 1 \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{K(1 - \cos 2w)}{w}$$

Example :-

Find Fourier Cosine and Sine transform of exponential function $f(x) = e^{-2x}$.

Solution :-

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x} \cos\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-2x}}{4+\omega^2} (\omega \sin\omega x - 2\cos\omega x) \right]_0^\infty$$

$$= \frac{1}{4+\omega^2} \sqrt{\frac{2}{\pi}} \left[e^{-2x} (\omega \sin\omega x - 2\cos\omega x) \right]_0^\infty$$

$$= \frac{1}{4+\omega^2} \sqrt{\frac{2}{\pi}} \left[0 - 1(-2) \right]$$

$$= \frac{2\sqrt{2}}{\sqrt{\pi}(4+\omega^2)}$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x} \sin\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-2x}}{4+\omega^2} (-2\sin\omega x - \omega\cos\omega x) \right]_0^\infty$$

$$= \frac{1}{4+\omega^2} \sqrt{\frac{2}{\pi}} \left[e^{-2x} (-2\sin\omega x - \omega\cos\omega x) \right]_0^\infty$$

$$= \frac{1}{4+\omega^2} \sqrt{\frac{2}{\pi}} \left[0 - 1(0 - \omega) \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{\omega}{4+\omega^2}$$

Notation :-

$F_c(f) = \{\hat{f}_c(\omega)\}$ = Fourier Cosine Transform of f.

$F_s(f) = \hat{f}_s(\omega)$ = Fourier Sine Transform of f.

Linearity :-

→ Fourier cosine transform is Linear transform.

Proof :-

$$\hat{F}_c(af+bg) = \sqrt{\frac{2}{\pi}}\int_0^\infty [a f(x) + b g(x)] \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[a\int_0^\infty f(x)\cos wx \, dx + b\int_0^\infty g(x)\cos wx \, dx\right]$$

$$= a\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx \, dx + b\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos wx \, dx$$

$$= a\hat{f}_c(w) + b\hat{g}_c(w)$$

$$\Rightarrow \hat{F}_c(af+bg) = a\hat{f}_c(w) + b\hat{g}_c(w)$$ (Proved)

→ Fourier Sine transform is Linear transform.

Proof :-

$$F_s(af+bg) = \sqrt{\frac{2}{\pi}}\int_0^\infty [af(x)+bg(x)] \sin wx \, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[a\int_0^\infty f(x)\sin wx \, dx + b\int_0^\infty g(x)\sin wx \, dx\right.$$

$$= a\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx \, dx + b\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\sin wx \, dx$$

$$= a\hat{f}_c(w) + b\hat{g}_c(w)$$

$$\Rightarrow F_s(af+bg) = aF_c(f) + bF_c(g)$$. (Proved)

Cosine and Sine Transforms of Derivatives :

Let $f(x)$ be continuous and absolutely integrable on the X-axis, let $f'(x)$ be piecewise continuous on each finite interval, and let $f(x) \to 0$ as $x \to \infty$,

Then,

$$F_c\{f'(x)\} = wF_s\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f(0)$$

$$F_c\{f'(x)\} = -wF_c\{f(x)\}$$

Proof :

$$F_c\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \underset{2^{nd}}{f'(x)}\, \underset{1^{st}}{\cos wx}\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[f(x)\cos wx \Big|_0^\infty + w\int_0^\infty f(x)\sin wx\, dx \right]$$

$$= -\sqrt{\frac{2}{\pi}}\, f(0) + w\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin wx\, dx$$

$$= -\sqrt{\frac{2}{\pi}}\, f(0) + wF_s\{f(x)\}$$

$$\Rightarrow \boxed{F_c\{f'(x)\} = wF_s\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f(0)} \text{ proved}$$

$$F_s\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x)\sin wx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[f(x)\sin wx \Big|_0^\infty - w\int_0^\infty f(x)\cos wx\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - w\int_0^\infty f(x)\cos wx\, dx \right]$$

$$= -w\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos wx\, dx$$

$$\Rightarrow \boxed{F_s\{f'(x)\} = -wF_c(f(x))} \text{ (Proved)}$$

$$\boxed{\begin{aligned} F_c\{f''(x)\} &= -\omega^2 F_c\{f(x)\} - \sqrt{\tfrac{2}{\pi}}\, f'(0) \\ F_s\{f''(x)\} &= -\omega^2 F_s\{f(x)\} + \sqrt{\tfrac{2}{\pi}}\, \omega f(0) \end{aligned}}$$

Problem Set :

Find Fourier cosine Transform

1. $f(x) = \begin{cases} 1, & 0 < x < 1 \\ -1, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

2. $f(x) = \begin{cases} x, & 0 < x < a \\ 0, & x > a \end{cases}$

3. $f(x) = \begin{cases} e^{x}, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} x^2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

Find Fourier Sine Transform

5. $f(x) = e^{-ax}, \quad a > 0, \; x > 0$

6. $f(x) = \begin{cases} x^2, & 0 < x < 1 \\ x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$