Oscillation and waves

Periodic motion: Any motion that repeats itself at a regular interval of time on the same path is known as periodic motion. e.g. the motion of earth around the sun, the motion of the hour-hand and the minute hand of a clock are periodic motion.

Simple harmonic motion:

Simple harmonic motion is a periodic motion in which the acceleration of the particle is directly proportional to the displacement from the mean position of rest and is always directed towards it.

characteristics of SHM

1) The motion is periodic and oscillatory.
2) The force and hence the acceleration of the particle executing SHM at any instant is proportional to the displacement measured form the mean position of the path.
3) The force and hence acceleration of the particle executing SHM at any instant is directed towards the motion of the path.

All SHM are periodic but all periodic motion are not SHM.

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Various Terms used in the description of a wave:

Suppose that the displacement of a particle at the left end of the string ($x=0$), where the wave originates is given by

$$y(x=0,t) = A\cos\omega t = A\cos 2\pi\nu t \quad [\because \omega = 2\pi\nu]$$

This is the particle oscillates in SHM with amplitude A, frequency ν, and angular frequency $\omega = 2\pi\nu$. The wave disturbance travels from $x=0$ to some point x to the right of the origin in an amount of time given by x/v, where v is the wave speed. So the motion of point x at time t is the same as the motion of point $x=0$ at the earlier time $t - x/v$. Hence we can find the displacement of point 'x' at time 't' by

$$y(x,t) = A\cos[\omega(t - x/v)]$$

$$= A\cos[\omega(x/v - t)] \quad \text{—①}$$

$[\because \cos(-\theta) = \cos\theta$; even function$]$

Again the wave number $K = \frac{2\pi}{\lambda}$

and $\lambda = \frac{v}{\nu} = \frac{2\pi v}{\omega}$ and $T = \frac{1}{\nu}$

So $$y(x,t) = A\cos\left(\frac{x\omega}{v} - \omega t\right) \quad \text{—②}$$

$$= A\cos\left(\frac{x\,2\pi\nu}{\lambda\nu} - 2\pi\nu t\right)$$

$$= A\cos\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \quad \text{—③}$$

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Again from (3)

$$y(x,t) = A\cos\left[\frac{2\pi x}{\lambda} - 2\pi\nu t\right]$$

$$\boxed{y(x,t) = A\cos(kx - \omega t)}$$

A -- amplitude

k - wave number - $\frac{2\pi}{\lambda}$

x - position

ω - angular frequency = $2\pi\nu$

Differential Equation of SHM

If a particle be displaced from its equlibrium position and caused to move uniformly along x-axis, then its under the action of an inertial force $m\frac{d^2x}{dt^2}$, where 'm' is the mass of the particle and $\frac{d^2x}{dt^2}$ is the acceleration at a distance x from the mean position.

over and above, a restoring force or the force of restitution acts on the particle given by

$$F \propto -x$$

or $F = kx$

which tends to bring back the particle to its position of equlibrium. Here k is the restoring force per unit displacement and called the stiffness constant. Under the dynamic equlibrium of the system

$$m\frac{d^2x}{dt^2} = -Kx$$

or $$\frac{d^2x}{dt^2} + \frac{K}{m}x = 0 \quad \text{------} \quad (0)$$

let $\frac{K}{m} = \omega_0^2$

or $$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \quad \text{--} \quad (1)$$

Equation (1) basically called the differential equation of SHM

As a trial solution let $x = Be^{pt}$

$\therefore \frac{dx}{dt} = pBe^{pt}$ and $\frac{d^2x}{dt^2} = p^2Be^{pt}$ — (2)

Substituting (2) in (1)

$$p^2Be^{pt} + \omega_0^2 Be^{pt} = 0$$

or $$Be^{pt}(p^2 + \omega_0^2) = 0$$

Since $Be^{pt} \neq 0$ being the displacement x

$$p^2 + \omega_0^2 = 0 \qquad [\because i = \sqrt{-1}]$$

or $$p = \pm i\omega_0$$

$\left[\because \begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{array}\right]$

So the general solution

$$x = B_1 e^{i\omega_0 t} + B_2 e^{-i\omega_0 t}$$

$$= B_1(\cos\omega_0 t + i\sin\omega_0 t) + B_2(\cos\omega_0 t - i\sin\omega_0 t)$$

$$= (B_1 + B_2)\cos\omega_0 t + i(B_1 - B_2)\sin\omega_0 t$$

— (3)

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Let $c_1 = \cancel{A_1 + A_2}\, B_1 + B_2$ and $c_2 = i(B_1 - B_2)$

So $x = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$

Putting $c_1 = A\sin\phi$ and $c_2 = A\cos\phi$

$$\phi = \tan^{-1}\left(\frac{c_1}{c_2}\right)$$

$\therefore$ $x = A\sin\phi\cos\omega_0 t + A\cos\phi\sin\omega_0 t$

$$\boxed{x = A\sin(\omega_0 t + \phi)} \quad \text{---} \ (3)$$

If $\phi = \theta + \pi/2$

$$\boxed{x = A\cos(\omega_0 t + \theta)} \quad \text{---} \ (4)$$

Here it may be noted that, the motion is repeated after an interval $\frac{2\pi}{\omega}$, Known as time period T.

$$\omega_0 = \sqrt{\frac{K}{m}}$$

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{K}}$$

frequency $\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$ $\quad \because \nu = \frac{1}{T}$

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Total Energy of a Linear Harmonic oscillator

If x is the displacement of the particle executing SHM at time 't' then the restoring force $F = -Kx$, If the particle is displaced further distance dx, then the work

$$dW = F dx = Kx\,dx \qquad [\because |F| = Kx]$$

This work remains stored in the particle in the form of potential energy (E_p)

$$E_p = \int_0^x Kx\,dx = \frac{1}{2}Kx^2$$

Kinetic energy $E_K = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$

Total energy $E = E_K + E_p = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}Kx^2$

In the absence of any dissipation, the total energy E remains constant and the system is called a conservative system. So for a conservative system $\frac{dE}{dt} = 0$

$$\text{or} \quad \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}Kx^2\right] = 0$$

$$\text{or} \quad m\frac{dx}{dt}\frac{d^2x}{dt^2} + Kx\frac{dx}{dt} = 0$$

$$\text{or} \quad m\frac{d^2x}{dt^2} + Kx = 0 \qquad \because \frac{dx}{dt} \neq 0$$

$$\frac{d^2x}{dt^2} + \frac{K}{m}x = 0$$

$$\text{or} \quad \frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

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Average Kinetic energy

Let $x = A\cos(\omega_0 t + \theta)$

velocity $v = \frac{dx}{dt} = -A\omega\sin(\omega t + \theta)$

$\therefore \quad E_K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega t + \theta)$

Maximum KE is $E_{km} = \frac{1}{2}m\omega_0^2 A^2$

The instataneous potential energy is

$E_p = \frac{1}{2}KA^2\cos^2(\omega_0 t + \theta) \qquad \because K = m\omega_0^2$

$E_p = \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega_0 t + \theta)$

Maximum PE

$E_{pm} = \frac{1}{2}m\omega_0^2 A^2$

The average KE of the particle is

$$\langle E_K \rangle = \frac{1}{T}\left[\frac{1}{2}m\int_0^T \left(\frac{dx}{dt}\right)^2 dt\right]$$

$$= \frac{m}{2T}\omega_0^2 A^2 \int_0^T \sin^2(\omega_0 t + \theta)\,dt$$

$$= \frac{1}{4}m\omega_0^2 A^2 = \frac{1}{2}E_{km}$$

The average PE

$$\langle E_p \rangle = \frac{1}{T}\left[\frac{1}{2}K\int_0^T x^2\,dt\right]$$

$$= \frac{KA^2}{2T}\int_0^T \cos^2(\omega_0 t + \theta)\,dt$$

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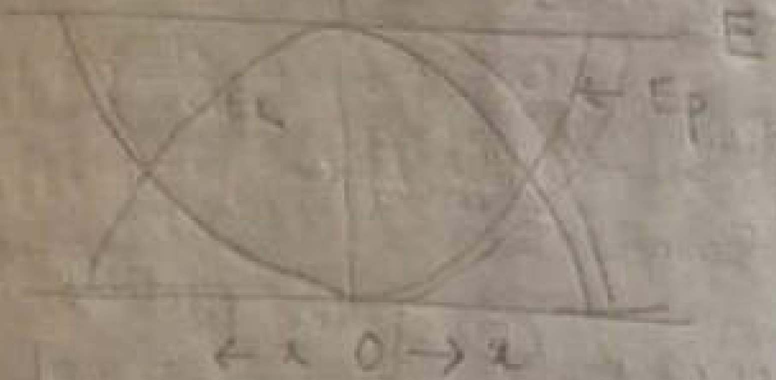
$$= \frac{1}{4} m\omega_0^2 A^2 = \frac{1}{2} E_{pm}$$

$$\langle E_k \rangle = \langle E_p \rangle = \frac{1}{2} E_{pm}$$

i.e. the average KE and average PE of the particle and half the corresponding maximum energy E_{km} and E_{pm} respectively

$$E = E_k + E_p = \frac{1}{2} m\omega_0^2 A^2$$



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Damped Harmonic Oscillator:

Free vibration: If a body is set into vibration by applying a force and left itself then after removing the force, the body continues to oscillate for ever with constant frequency and constant amplitude. The frequency of the body is determined by the inertial and elastic properties of the body and is called the natural frequency. This type of SHM which persists indefinetely without loss of amplitude is called free or undamped vibration.

Damped vibration: All oscillations experience damping force such as friction and resistance of the medium. As a result, the energy and amplitude of the oscillator decreases continuously and eventually the oscillation stops. Such oscillations whose amplitude, in successive oscillations decreasing due to the presence of resistive force are called damped oscillations. e.g. oscillating pendulum in air or vibrating tuning fork.

Let a damped harmonic oscillator having one degrees of freedom be at a distance 'x' from the equlibrium position during motion at any instant of time.

The force acting on the system are

a) Force of restitution, i.e. the force tends to restore the system to its original equlibrium position and is proportional to the displacement 'x' and is given by $-Kx$, where K is the restoring force per unit displacement.

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(b) Damping force : This force acting on the oscillatory system causes dissipation of energy and is given by $-b\frac{dx}{dt}$, where b is damping force per unit velocity.

$$F_{net} = F_{res} + F_{dam}$$

$$m\frac{d^2x}{dt^2} = -Kx - b\frac{dx}{dt}$$

or $$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{K}{m}x = 0$$

Let $\frac{b}{m} = 2\gamma$ and $\frac{K}{m} = \omega_0^2$

So $$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0 \quad \text{--- (1)}$$

Let us suppose the trial solution be of the form

$$x(t) = Be^{\alpha t}$$

$$\left.\begin{aligned} \frac{dx}{dt} &= B\alpha e^{\alpha t} = \alpha x \\ \frac{d^2x}{dt^2} &= B\alpha^2 e^{\alpha t} = \alpha^2 x \end{aligned}\right\} \quad \text{--- (2)}$$

Putting (2) in (1)

$$(\alpha^2 + 2\gamma\alpha + \omega_0^2)x = 0$$

Since $x \neq 0$ being the displacement

$$\alpha^2 + 2\gamma\alpha + \omega_0^2 = 0$$

or $$\alpha = \cancel{-\gamma}\ -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

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α has two roots

$\alpha_1 = -\gamma + \sqrt{\gamma^2 - \omega_0^2}$ and $\alpha_2 = -\gamma - \sqrt{\gamma^2 - \omega_0^2}$

so the general solution of ①

$$x(t) = B_1 e^{\alpha_1 t} + B_2 e^{\alpha_2 t}$$

$$= B_1 e^{(-\gamma + \sqrt{\gamma^2 - \omega_0^2})t} + B_2 e^{(-\gamma - \sqrt{\gamma^2 - \omega_0^2})t}$$

$$x(t) = e^{-\gamma t}\left[B_1 e^{\sqrt{\gamma^2 - \omega_0^2}\,t} + B_2 e^{-\sqrt{\gamma^2 - \omega_0^2}\,t}\right] \quad \text{—③}$$

where the constants B_1 and B_2 depends on the initial (at $t=0$) position and velocity of the oscillator.

The behaviour of the damped oscillator depends on the relative value of the restoring force and the damping force which regulate the motion.

Case A : $\gamma > \omega_0$, The term $\sqrt{\gamma^2 - \omega_0^2}$ is real and the magnitude will be less than γ, therefore both the exponents $(-\gamma + \sqrt{\gamma^2 - \omega_0^2})$ and $(-\gamma - \sqrt{\gamma^2 - \omega_0^2})$ in equation ③ are negative. Due to this reason, the displacement 'x' continuously decreases exponentially to zero without performing any oscillation. This kind of motion is known as overdamped.

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Case B $\gamma < \omega_0$, the term $\sqrt{\gamma^2 - \omega_0^2}$ is imaginary, which can be written as

$$\sqrt{\gamma^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \gamma^2} = i\beta$$

where $\beta = \sqrt{\omega_0^2 - \gamma^2}$ and $i = \sqrt{-1}$

Equation 3 becomes

$$x(t) = e^{-\gamma t}\left[B_1 e^{i\beta t} + B_2 e^{-i\beta t}\right]$$

$$= e^{-\gamma t}\left[B_1(\cos\beta t + i\sin\beta t) + B_2(\cos\beta t - i\sin\beta t)\right]$$

$$= e^{-\gamma t}\left[(B_1 + B_2)\cos\beta t + i(B_1 - B_2)\sin\beta t\right]$$

Let $B_1 + B_2 = A\sin\delta$ and $i(B_1 - B_2) = A\cos\delta$

$$= e^{-\gamma t}\left[A\sin\delta\cos\beta t + A\cos\delta\sin\beta t\right]$$

$$= Ae^{-\gamma t}\sin(\beta t + \delta)$$

$$= Ae^{-\gamma t}\sin\left[\sqrt{\omega_0^2 - \gamma^2}\, t + \delta\right] \quad \text{--- (4)}$$

The above equation shows the oscillatory motion and represents the damped harmonic oscillator. The oscillations are not simple harmonic because the amplitude ($Ae^{-\gamma t}$) is not constant but decreases with time (t). However the decay of the amplitude depends upon the damping factor γ, the motion is known as underdamped motion.

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Case C $\gamma = \omega_0$. In this case equation ③ does not satisfy equation ①. Suppose $\sqrt{\gamma^2-\omega_0^2}$ is not exactly zero but it is equal to a very small quantity μ. Now equation ③ gives

$$x(t) = e^{-\gamma t}\left[B_1 e^{\mu t} + B_2 e^{-\mu t}\right]$$

$$= e^{-\gamma t}\left[B_1(1+\mu t+\cdots) + B_2(1-\mu t+\cdots)\right]$$

As μ is small, neglecting higher order terms

$$\left[\text{Hint: } e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right.$$

$$\left. e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \right]$$

$$= e^{-\gamma t}\left[(B_1+B_2) + \mu t(B_1-B_2)\right]$$

$$x(t) = e^{-\gamma t}\left[P' + Q' t\right] \quad \text{— ⑤}$$

where $P' = B_1 + B_2$ & $Q' = \mu(B_1 - B_2)$

It is clear that 't' increases the term $(P'+Q't)$ but $e^{-\gamma t}$ gets decreased. Because of this fact, the displacement 'x' first increases due to the term $(P'+Q't)$, but it decreases due to the exponential term $e^{-\gamma t}$ and finally approaches to zero as 't' increases. This type of motion is called critically damped motion.

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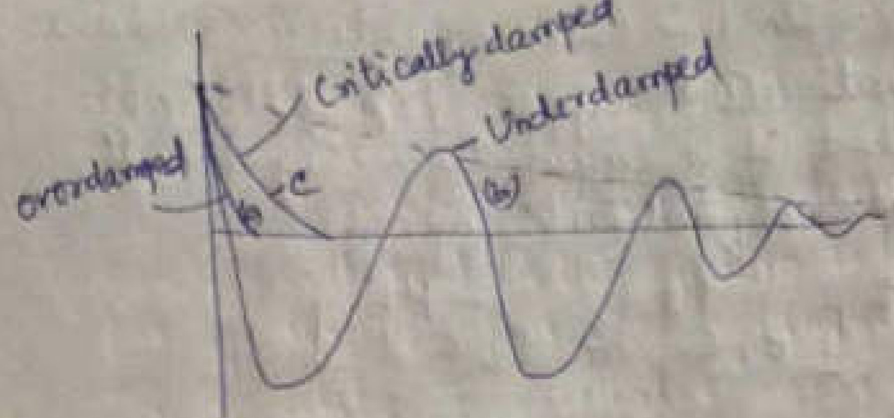

Logarithmic Decrement:

The rate at which the amplitude dies away is measured by logarithmic decrement. The amplitude of the damped harmonic oscillator is given by the factor $e^{-\gamma t}$. Therefore, at $t=0$, the amplitude will be maximum (i.e. $A=A_0$). If $A_1, A_2 \ldots$ be the amplitude at time $t=T, 2T, \ldots$ respectively, where T is the time period of oscillations, then

$$A_1 = Ae^{-\gamma T}, \quad A_2 = Ae^{+\gamma(2T)}$$

$$\frac{A_0}{A_1} = \frac{A_1}{A_2} = \cdots = e^{+\gamma T} = e^{\lambda}$$

Here, λ is called logarithmic decrement

$$\ln\frac{A_0}{A_1} = \ln\frac{A_1}{A_2} = \cdots = \lambda$$

Hence, logarithmic decrement is the natural logarithm of ratio between two successive maximum amplitudes, which are separated by one period.

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Relaxation time:

It is the time taken by the damped harmonic oscillator for decaying total mechanical energy by the factor $\frac{1}{e}$ of its initial value.

The mechanical energy of a damped harmonic oscillator is

$$E = \frac{1}{2} m A^2 \omega_0^2 e^{-2\gamma t} \quad - (1)$$

$$E = E_0 e^{-2\gamma t} \quad - (2) \quad \because E_0 = \frac{1}{2} m A^2 \omega_0^2$$

Suppose 'τ' be the relaxation time, then at $t = \tau$

$$E = \frac{E_0}{e}$$

From (2) $$\frac{E_0}{e} = E_0 e^{-2\gamma\tau}$$

$$e^{-1} = e^{-2\gamma\tau}$$

$$e = e^{2\gamma\tau}$$

$$\boxed{\tau = \frac{1}{2\gamma}}$$

Therefore, the dissipated energy in terms of relaxation time is written as

$$E = E_0 e^{-t/\tau}$$

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Quality Factor:

It is defined as 2π times the ratio of energy stored in the system to the energy lost per cycle. This factor of a damped oscillator shows the quality of oscillator so far as damping is concerned.

$$Q = 2\pi \times \frac{\text{average energy stored in one cycle}}{\text{average energy lost in one period}}$$

$$= 2\pi \times \frac{E}{P_d T}$$

when P_d is the power dissipation & T is the periodic time

$$Q = 2\pi \frac{E}{(E/\tau)T} = \frac{2\pi\tau}{T}$$

$\left[\because P_d = \frac{E}{\tau}\right.$

$\omega = \frac{2\pi}{T}$

$$\boxed{Q = \omega\tau}$$

For the force constant K and mass m of the vibrating system

$$\omega = \sqrt{\frac{K}{m}} \quad \& \quad \tau = \frac{1}{2\gamma}$$

$$Q = \frac{1}{2\gamma}\sqrt{\frac{K}{m}}$$

Since lower value of γ lead to lower damping, it is clear that for low damping, the Quality factor would be higher.

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Forced vibrations:

If a body placed in an external force while it is vibrating is known as forced vibrations. For example, If a bob of simple pendulum is held in hand and given numbers of swings by hand. In this case, the pendulum vibrates due to external force and not due to the natural frequency. So forced vibrations can also be defined as the vibrations which the body vibrates with frequency other than its natural frequency, which is due to some external periodic force.

Theory of forced vibrations:

Suppose a particle of mass 'm' is connected to a spring, when it is displaced from its mean position, oscillations are started and the particle different kinds of forces, a restoring force $(-Kx)$, a damping force $(-b\frac{dx}{dt})$ and the external periodic force $F_0 \sin\omega t$. The total force acting on the particle is, therefore

$$F = F_0 \sin\omega t - b\frac{dx}{dt} - Kx \quad --- (1)$$

by Newton's second law of motion

$$F = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} = F_0 \sin\omega t - b\frac{dx}{dt} - Kx$$

or $$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{K}{m}x = \frac{F_0 \sin\omega t}{m} \quad - (2)$$

Substituting $\frac{b}{m} = 2r$ & $\frac{K}{m} = \omega_0^2$ & $\frac{F_0}{m} = f$

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$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = f \sin\omega t \quad --- (3)$$

In the steady state, the solution of the above equation should be

$$x = A \sin(\omega t - \delta) \quad --- (4)$$

when A is the amplitude of vibrations in the steady state.

So $\frac{dx}{dt} = \omega A \cos(\omega t - \delta)$

$\frac{d^2x}{dt^2} = -\omega^2 A \sin(\omega t - \delta)$

Substituting in equation (3)

$$-\omega^2 A \sin(\omega t-\delta) + 2\gamma\omega A \cos(\omega t - \delta) + \omega_0^2 A \sin(\omega t-\delta)$$
$$= f \sin\{(\omega t - \delta) + \delta\}$$

or $$A(\omega_0^2 - \omega^2)\sin(\omega t - \delta) + 2\gamma\omega A \cos(\omega t - \delta)$$

$$= f \sin(\omega t-\delta)\cos\delta + f\cos(\omega t-\delta)\sin\delta \quad -(5)$$

If equation (5) holds for all values of 't', then the co-efficients of $\sin(\omega t - \delta)$ and $\cos(\omega t-\delta)$ must be equal on both sides, then

$$A(\omega_0^2 - \omega^2) = f\cos\delta \quad — (6)$$

$$2\gamma\omega A = f \sin\delta \quad — (7)$$

By squaring and adding (6) & (7)

$$A^2(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2A^2 = f^2$$

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$$\boxed{A = \frac{f}{\sqrt{(\omega_0^2-\omega^2)^2+4r^2\omega^2}}} \quad -⑧$$

Dividing ⑦ by ⑥

$$\tan\delta = \frac{2r\omega}{\omega_0^2-\omega^2}$$

$$\text{or} \quad \boxed{\delta = \tan^{-1}\left[\frac{2r\omega}{\omega_0^2-\omega^2}\right]} \quad -⑨$$

From equation ⑧ and ⑨ it is clear that, the amplitude and phase of forced oscillations depend upon $(\omega_0^2-\omega^2)$ i.e. the driving frequency (ω) and the natural frequency (ω_0) of the oscillator. The amplitude & phase are explained as below.

Case-A : Very low driving frequency, i.e. $\omega \ll \omega_0$

$$\text{Hence} \quad A = \frac{f}{\sqrt{(\omega_0^2-\omega^2)^2+4r^2\omega^2}} \simeq \frac{f}{\omega_0^2}$$

$$= \frac{F_0}{m\omega_0^2} \qquad \because f = \frac{F_0}{m}$$

$$A = \frac{F_0}{K} \qquad \because K = m\omega_0^2$$

Hence, the amplitude depends upon the force constant of the spring and the magnitude of applied force.

$$\text{phase } \delta = \tan^{-1}\left[\frac{2r\omega}{\omega_0^2-\omega^2}\right] \simeq \tan^{-1}\left[\frac{2r\omega}{\omega_0^2}\right]$$

since $\omega_0^2 >> \omega$, $2r\omega/\omega_0 \to 0$ & $\delta \to 0$ or $\simeq 0$

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Therefore, under this situation the driving force and the displacement are in phase.

Case B : $\omega = \omega_0$, i.e., same driving and natural frequencies. This frequency is called the ~~[illegible]~~ resonance frequency

$$A = \frac{f}{\sqrt{(\omega_0^2-\omega^2)^2 + 4r^2\omega^2}} \simeq \frac{f}{2r\omega}$$

$$= \frac{F_0/m}{\frac{b}{m}\omega} = \frac{F_0}{b\omega}$$

Hence the amplitude of vibrations depends upon the damping and applied force. Now

$$\delta = \tan^{-1}\left[\frac{2r\omega}{(\omega_0^2-\omega^2)}\right] = \tan^{-1}\left[\frac{2r\omega}{0}\right]$$

$$= \tan^{-1}[\infty] = \frac{\pi}{2}$$

Thus, the displacement lags behind the force by a phase of $\frac{\pi}{2}$, as $x = A\sin(\omega t - \delta)$ and the applied force is $F_0 \sin \omega t$.

Case C : Very large driving frequency, i.e. $\omega >> \omega_0$,

$$A = \frac{f}{\sqrt{(\omega_0^2-\omega^2)^2 + 4r^2\omega^2}}$$

since $\omega^4 >> 4\omega^2 r^2$

$$A = \frac{f}{\omega^2} = \frac{F_0}{m\omega^2}$$

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$$\delta = \tan^{-1}\left[\frac{2\gamma\omega}{(\omega_0^2-\omega^2)}\right] = \tan^{-1}\left[\frac{2\gamma\omega}{-\omega^2}\right]$$

$$\simeq \tan^{-1}\left[\frac{2\gamma}{-\omega}\right] = \tan^{-1}[-0] = \pi$$

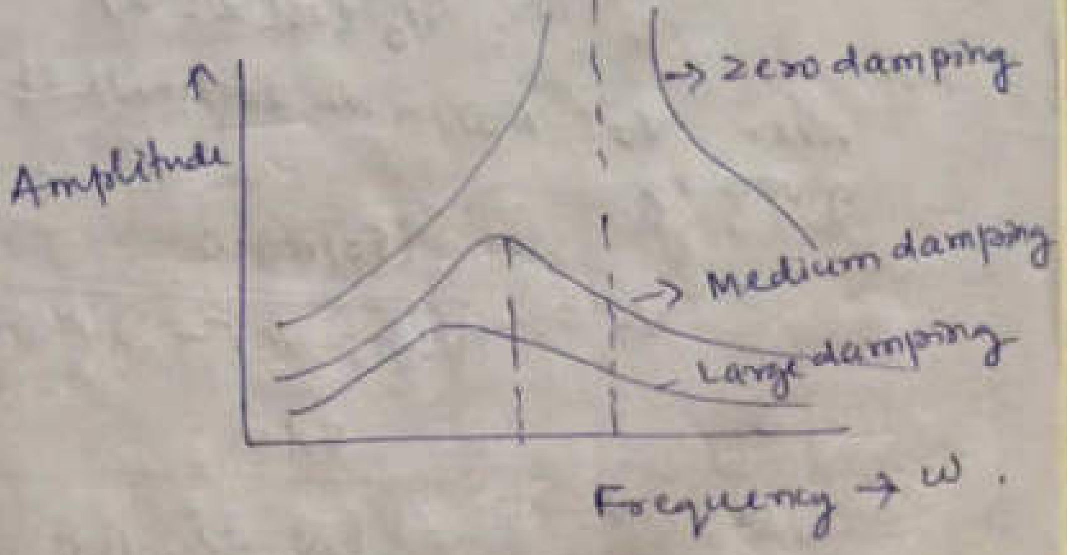
Therefore, under the situation $\omega >> \omega_0$, the displacement lags behind the force by a phase of π.

Resonance:

From equation ⑧ it is clear that the amplitude of driven oscillator is proportional to the amplitude F of the driving force and depends on ω & ω_0. The amplitude attains maximum value at $\omega = \omega_0$

$$A_{max} = \frac{F_0}{b\omega}$$

At $\omega = \omega_R$ the oscillator is said to be resonate with the driving force. This phenomenon is called resonance. The frequency $\nu_R = \frac{\omega_R}{2\pi}$ is called resonant frequency.



Amplitude Resonance

When the displacement amplitude of forced vibration is maximum for a particular frequency of applied periodic force, the phenomenon is called amplitude resonance.

The amplitude of forced oscillation:

$$A = \frac{f}{\sqrt{(\omega_0^2-\omega^2) + 4r^2\omega^2}}$$

Clearly A varies with ω; A is maximum when denominator is minimum

$$\therefore \frac{d}{d\omega}\left[\sqrt{(\omega_0^2-\omega^2)^2 + 4r^2\omega^2}\right] = 0$$

$$2(\omega_0^2-\omega^2)(-2\omega) + 8r^2\omega = 0$$

$$-4\omega\left[(\omega_0^2-\omega^2) - 2r^2\right] = 0$$

Since $\omega \neq 0$

$$\omega^2 = \omega_0^2 - 2r^2$$

or $\omega = \sqrt{\omega_0^2 - 2r^2} = \omega_r \quad [\because \omega_0^2 > 2r^2]$

cyclic periodic frequency $f_r = \frac{\sqrt{\omega_0^2 - 2r^2}}{2\pi}$ for $A = A_{max}$

Sharpness of resonance:

Rate of fall in amplitude with the change of the frequency of applied periodic force for low K, $\omega \approx \omega_0$, $A_{max} = \frac{f}{2r\omega}$

for small K, Curve is sharper

for large K, Curve is flatter.

Resonance.

It is the phenomenon of setting a body onto vibration by the application of a strong periodic force such that the frequency of the impressed force coincides with the natural frequency of the body.

Analytical treatment for Resonance.

For resonance to take place, the amplitude of forced vibration in the steady state to be maximum

$$\text{ie} \quad A = \frac{f}{\sqrt{(\omega_0^2-\omega^2)^2+4r^2\omega^2}} \text{ should be } \text{~~minimum~~ maximum}$$

$$\text{or} \quad \sqrt{(\omega_0^2-\omega^2)^2+4r^2\omega^2} \text{ is to be minimum.}$$

$$\text{or} \quad \frac{d}{d\omega}\left[\sqrt{(\omega_0^2-\omega^2)^2+4r^2\omega^2}\right] = 0$$

$$2(\omega_0^2-\omega^2)(-2\omega) + 8r^2\omega = 0$$

$$(\omega_0^2-\omega^2)(-4\omega) + 8r^2\omega = 0$$

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dividing (-4ω) with each term

$$(\omega_0^2-\omega^2) - 2r^2 = 0$$

$$\omega^2 = \omega_0^2 - 2r^2$$

or $$\omega_R = \cancel{\omega_0^2 - 2r^2} \sqrt{\omega_0^2 - 2r^2}$$

Hence for resonance; amplitude of vibration to be maximum, the angular frequency should be $\sqrt{\omega_0^2 - 2r^2}$. This is necessary and sufficient condition for resonance.

<u>Resonance frequency</u>:

$$\omega_R = \sqrt{\omega_0^2 - 2r^2}$$

or $$\nu_R = \frac{\sqrt{\omega_0^2 - 2r^2}}{2\pi}$$

<u>Resonance Amplitude</u>:

The maximum amplitude for which resonance comes into existence is called resonance amplitude

$$A = \frac{f}{\sqrt{(\omega_0^2-\omega_R^2)^2 + 4r^2\omega_R^2}}$$

$$\boxed{A_{Res} = \frac{f}{2r\sqrt{\omega_0^2 - r^2}}}$$

$$= \frac{F_0}{2m r\sqrt{\omega_0^2 - r^2}}$$

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For small damping i.e. $\gamma^2 << \omega_0^2$

$$A_{res} = \frac{F_0}{2m\gamma\omega_0} = \frac{f}{b\omega_0}.$$

resonance amplitude varies with the damping constant.

<u>Sharpness of resonance.</u>

The amplitude of forced oscillator is

$$A = \frac{f}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$

For amplitude to be maximum, i.e. for the occurance of resonance $\omega=\omega_0$. As the frequency of the applied force(ω) increased or decreased from its resonant value (ω_0), the value of amplitude always decreases

⇒ when the amplitude falls rapidly for a small change of frequency (ω) of the applied force from the resonant value, the resonance is said to be sharp

⇒ when the amplitude falls slowly for a slight change of frequency (ω) of the applied force from the resonant value, the resonance is said to be flat.

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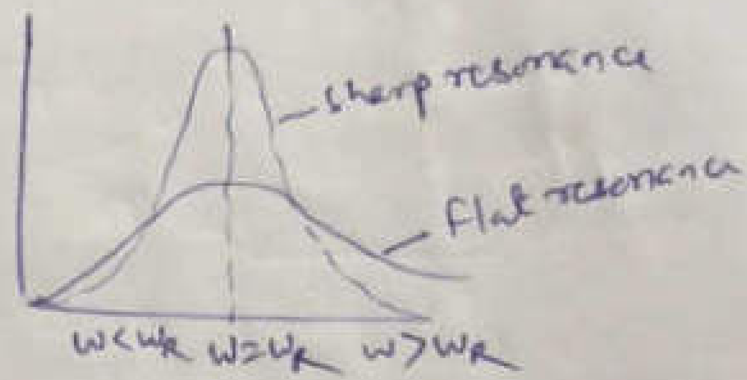

when the damping is small, resonance is sharp & when the damping is large, resonance is flat.

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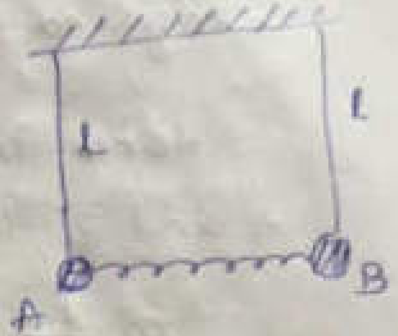
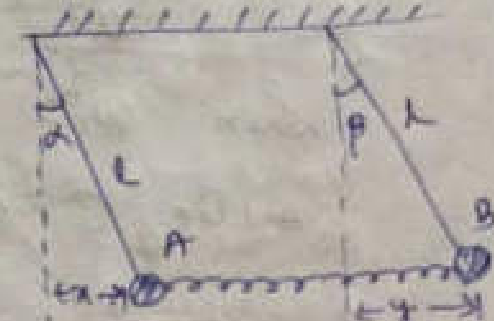
Coupled Oscillations:

(1) Coupled system of two pendulums:



Two identical pendulums A and B each having a identical bob of mass m, suspended by rigid, weightless rod of length L form a rigid support.

The two pendulums bobs are connected by a light spring of force constant K. The normal length of the spring is equal to the distance between the bobs when they are in equilibrium position. In this condition, the spring does not exert any force on the pendulum bobs. However, when the bobs undergo unequal displacements, the spring gets either stretched or compressed depending on the relative displacements of the bobs. The deformed spring exerts force on the bobs. The pendulum bobs are set into oscillation in the plane of the pendulum.



At a given instant, the displacements of the bobs A and B are 'x' & 'y' respectively (in the same direction). The restoring force, due to the spring on A and B are

$-K(x-y)$ & $-K(y-x)$ respectively. Similarly the restoring force on A and B due to the components of gravitational force are

$$-mg\sin\alpha = \frac{-mgx}{l} \quad \& \quad -mg\sin\beta = \frac{-mgy}{l}$$

Hence, the equation of motion of the pendulum A and B are

$$m\frac{d^2x}{dt^2} = \frac{-mgx}{l} - K(x-y) \quad \text{--- (1)}$$

$$\& \quad m\frac{d^2y}{dt^2} = \frac{-mgy}{l} - K(y-x) \quad \text{--- (2)}$$

Each of the above equations involves both x & y. Hence these are coupled oscillations.

The equation of motion given by (1) & (2) can be written in the form

$$\frac{d^2x}{dt^2} + \omega_1^2 x + \frac{k}{m}(x-y) = 0 \quad \text{--- (3)}$$

$$\frac{d^2y}{dt^2} + \omega_1^2 y + \frac{k}{m}(y-x) = 0 \quad \text{--- (4)}$$

when $\omega_1^2 = g/l$

Adding (3) & (4) and subtracting (4) from (3)

$$\frac{d^2}{dt^2}(x-y) + \left(\omega_1^2 + \frac{2k}{m}\right)(x-y) = 0 \quad \text{--- (5)}$$

$$\frac{d^2}{dt^2}(x+y) + \omega_1^2(x+y) = 0 \quad \text{--- (6)}$$

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Let $Q_1 = x + y$ & $Q_2 = x - y$

So

$$\frac{d^2Q_1}{dt^2} + \omega_1^2 Q_1 = 0 \quad \text{--- (7)}$$

$$\frac{d^2Q_2}{dt^2} + \omega_2^2 Q_2 = 0 \quad \text{--- (8)}$$

where $\omega_2^2 = \omega_1^2 + \frac{2k}{m} = \frac{g}{l} + \frac{2k}{m}$

The equations of motion (7) & (8) interms of coordinates Q_1 & Q_2 are decoupled & each equation describes the oscillation of a simple harmonic oscillator.

Normal Coordinates

The coordinates $Q_1 = x + y$ & $Q_2 = x - y$ are called normal coordinates of the couple system. The normal coordinates are linear combinations of the original variables x & y. The oscillations described interms of the normal coordinates are independent and are called normal modes of oscillation.

Normal Mode of frequencies

$$\omega_1 = \sqrt{g/l} \quad \& \quad \omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$$

The corresponding frequencies

$$\nu_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \quad \& \quad \nu_2 = \frac{1}{2\pi}\sqrt{\frac{g}{l} + \frac{2k}{m}}$$

are called normal mode of frequencies of the coupled oscillator.

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Normal Modes of oscillation:

By suitably choosing the initial condition it is always possible to describe the oscillation of the coupled oscillator interms of only one normal coordinate. The system oscillates with the corresponding normal mode frequency. The oscillation of the coupled system interms of normal coordinates is called normal mode of oscillation.

Q_1 - mode

If the initial condition are chosen such that $x = y$, i.e. both the pendulum bobs are displaced by the same amount in the same direction and $Q_2 = x - y = 0$. Thus only the Q_1 mode is excited and the equation of motion is described by only one equation. Since both the bobs are equally displaced the spring is always in the normal state and both the bobs oscillate with same amplitude, frequency and phase. This is called the in phase mode of oscillation.

Q_2 - mode.

If $x = -y$, i.e. the bobs are displaced by the same amount in opposite direction. $Q_1 = 0$. so, here only the Q_2 - mode is excited. The angular frequency is greater than that of individual oscillations. The bobs oscillate with opposite phase. This is called out-of-phase mode of oscillation. Since $\omega_2 > \omega_1$, the

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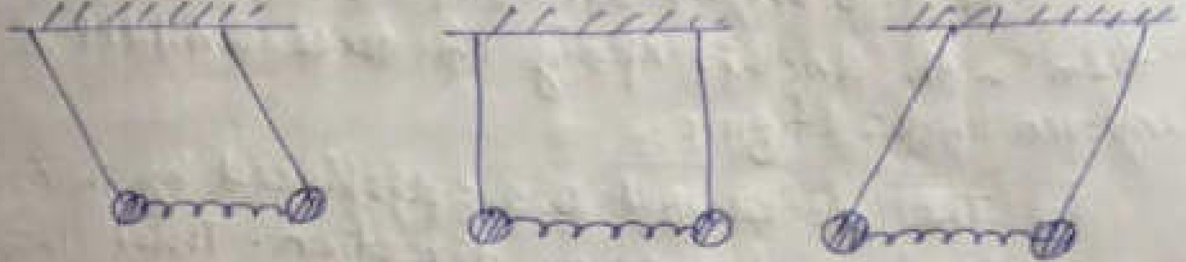
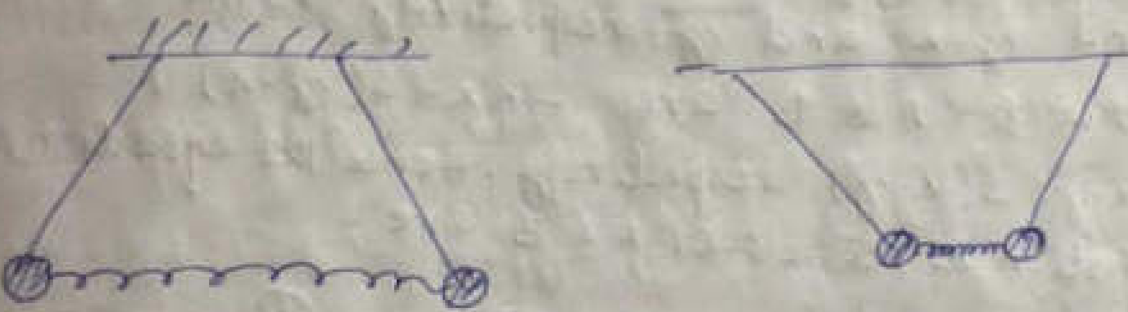
frequency of out-of-phase mode of oscillation is always greater than the frequency of in-phase mode.

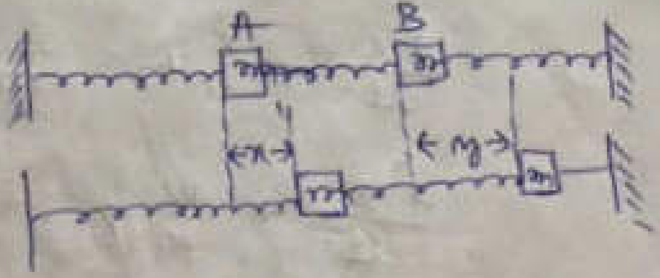


In-phase mode



Out-of-phase mode

② Coupled mass-spring System



Two identical blocks of mass m each are connected to two rigid supports by springs of force constant K, so that they can oscillate on a frictionless

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horizontal table with angular frequency

$$\omega_1 = \sqrt{\frac{K}{m}}$$

The two blocks are coupled together by joining them with another spring of force constant 's' as shown in the figure.

The blocks A & B are displaced by x & y, respectively in the same direction. Thus the 1^{st}, 2^{nd} and 3^{rd} springs are extended by x, $y-x$ and $y-x$ and y respectively. So the restoring force on A & B are $-Kx + s(y-x)$ & $-Ky + s(x-y)$ respectively. Hence the equation of motion of the blocks A & B are

$$m\frac{d^2x}{dt^2} = -Kx + s(y-x) \quad \text{--- (1)}$$

$$\& \quad m\frac{d^2y}{dt^2} = -Ky - s(y-x) \quad \text{--- (2)}$$

Equations (1) & (2) can be written as

$$\frac{d^2x}{dt^2} + \frac{K}{m}x - \frac{s}{m}(y-x) = 0 \quad \text{--- (3)}$$

$$\frac{d^2y}{dt^2} + \frac{K}{m}y + \frac{s}{m}(y-x) = 0 \quad \text{--- (4)}$$

Adding (3) & (4)

$$\frac{d^2}{dt^2}(x+y) + \frac{K}{m}(x+y) = 0$$

Subtracting (4) from (3)

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$$\frac{d^2}{dt^2}(x-y) + \left(\frac{k}{m}+\frac{2s}{m}\right)(x-y) = 0$$

let $Q_1 = x+y$ & $Q_2 = x-y$

$$\frac{d^2Q_1}{dt^2} + \omega_1^2 Q_1 = 0$$

& $$\frac{d^2Q_2}{dt^2} + \omega_2^2 Q_2 = 0$$

when $\omega_1^2 = \frac{k}{m}$ & $\omega_2^2 = \frac{k+2s}{m}$

The normal mode frequencies of the coupled mass spring system are

$$\nu_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \& \quad \nu_2 = \frac{\omega_2}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}+\frac{2s}{m}}$$

(i) If both blocks are displaced by the same amount on the same side, i.e. $x=y$, then $Q_2 = 0$, so Q_1 mode or in-phase mode of oscillation with frequency ν_1 is excited.

(ii) If $y=-x$ i.e. blocks are displaced by the same amount in opposite directions then $Q_1 = 0$, so out-of-phase mode or Q_2 mode is excited with frequency ν_2.

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Concept of wave and wave equation:

Waves can be thought as periodic variation of space and time that propagates. Let us suppose we shake a string form one end, the wave function $y(x,t)$ represents the displacement x at any instant 't'.

$$y(x,t) = A\cos(kx-\omega t)$$

The transverse velocity

$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = +\omega A\sin(kx-\omega t)$$

The acceleration

$$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A\cos(kx-\omega t)$$

$$= -\omega^2 y$$

The second partial derivative w.r.t x tells us the curvature of the string

$$\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 A\cos(kx-\omega t) = -k^2 y(x,t)$$

$$\omega = vk$$

$$\frac{\partial^2 y(x,t)/\partial t^2}{\partial^2 y(x,t)/\partial x^2} = \frac{\omega^2}{k^2} = v^2 \text{ and}$$

$$\boxed{\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}}$$

wave equation.

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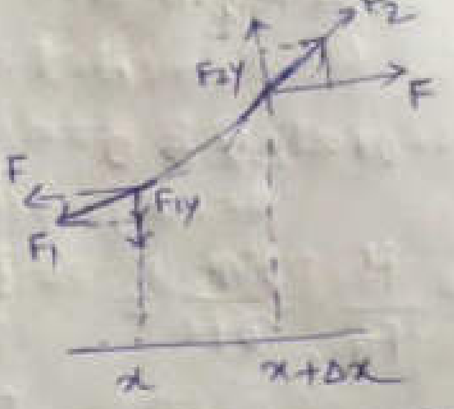
Another method



Let us consider a small portion of a string oscillating whose length is Δx. The mass of the segment is $m = \mu \Delta x$, where μ is the mass per unit length. The x-component of the forces have equal magnitude F and add to zero

$\frac{F_{1y}}{F}$ is equal to the ~~slope~~ magnitude to the slope of the string at x & $\frac{F_{2y}}{F}$ at $x+\Delta x$

$$\frac{F_{1y}}{F} = -\left(\frac{\partial y}{\partial x}\right)_x \quad \left(\frac{F_{2y}}{\partial x}\right)_{x+\Delta x} \quad —①$$

$$F = F_{1y} + F_{2y} = F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

According to Newton's second law

$$F = \mu \Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\mu \Delta x \frac{\partial^2 y}{\partial t^2} = F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] \quad ②$$

Dividing both side by $F \Delta x$

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$$\frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} = \frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x}$$

as $\Delta x \to 0$

$$\frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

Since $v = \sqrt{\frac{F}{\mu}}$

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

or $\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$ wave equation.

Longitudinal waves.

If the motion of the particles constituting a wave is along the direction of propagation of the wave, then the wave is a longitudinal wave.

ex. sound wave, slinky

Standing wave.

Let us consider two waves moving in opposite direction

$$y_1 = a\sin(\omega t - kx)$$

$$y_2 = a\sin(\omega t + kx)$$

$$y_R = y_1 + y_2$$

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$y_R = 2a\sin(kx)\cos(\omega t)$

The points of minimum amplitude are those for which (node)

$\sin(kx) = 0 \quad kx = 0, \pi, 2\pi \ldots$

$\quad$ & $x = \lambda/2, \lambda, 3\lambda/2, 2\lambda \ldots$

The points for maximum amplitude (anti node)

$\sin(kx) = 2a$ for $kx = \pi/2, \frac{3\pi}{2}, 5\pi/2$

$\quad$ or $x = \lambda/4, 3\lambda/4, 5\lambda/4$

Reflection and transmissions at boundary:

1) when Both ends of string is fixed

As a pulse travels along a stretched string and strikes at fixed end, it exerts a force and according to Newton's third law, it bounce back and the incident & reflected wave superimpose destructively to produce a point of zero displacement or nodal points at fixed ends.

For $x = 0$

$$y = 2a\sin(k \cdot 0)\cos(\omega t) \text{ at } x = 0$$

$$\& \ y = 2a\sin(k \cdot L)\cos(\omega t) \text{ at } x = L$$

$$\sin(k \cdot L) = 0 = \sin n\pi$$

$$\lambda = \lambda_n = \frac{2L}{n}, \quad n = 1, 2 \ldots$$

$$\nu_n = \frac{v}{\lambda_n} = \frac{nv}{2L} \quad \text{for } n = 1, 2, 3 \ldots$$

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Thus, a string that is fixed at both ends can vibrate with certain well defined frequencies. The fundamental mode is defined by n=1, wherein the value of wavelength is defined as 2L.

Hence, a reflection at a fixed end generates a reflected wave that undergoes a phase change of π.

2) <u>One end of the string is fixed:</u>

As a pulse travels along a stretched string, it passes through the free end, which can be represented by a light ring or pulley. Owing to the force exerted by the pulse on the ring of the pulley, it experiences an acceleration, which in turn introduces a reaction force on the string. This gives rise to a pulse that travels in a direction opposite to the incident wave in the string.

Hence there is a reflection at the free end as well, but it generates a reflected wave/pulse that does not undergo a phase change of π. The fixed end of the string has a node, whereas the free end has an antinode.

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