UNIT - IV

Scalar and vector Fields:

A contineous function of the position of a point in a region of space is called point function. The region of space in which it specifies a physical quantity is known as a field. These fields are classified into two groups

(i) Scalar field: A scalar field is defined as that region of space, whose each point is associated with a scalar point function, i.e; a contineous function which gives the value of a physical quantity as flux, potential, temperature, etc. In a scalar field, all the points having the same scalar physical quantity are connected by the means of surfaces called equal or level surfaces.

(ii) vector field: A vector field is specified by a contenuous vector point functions having magnitude and direction, both of which change from point to point, in the given region of field. The method of presentation of a vector field is called vector lines, or lines of surfaces.

Gradient of a scalar field

The gradient of a scalar point function $\phi(x,y,z)$ is defined as $\nabla\phi$ and is written as

$$\text{Grad}\,\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Grad ϕ is a vector quantity.

Interpretation:

Derivative of a function of one variable tells us how fast the function varies if we move.

a small distance. It means the gradient is the rate of change of a quantity with distance. For example, temperature gradient in a metal bar is the rate of change of temperature along the bar. However, for a function of three variables, the situation is more complicated, as it depends on what direction we chose to move. For a function of $\phi(x,y,z)$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

Here $d\phi$ is a measure of change in ϕ that occurs when we alter all three variables by small amounts dx, dy and dz.

$$d\phi = \vec{\nabla}\phi \cdot \vec{dl}$$

where $\vec{\nabla}\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ is the gradient of ϕ. Gradient is a vector quantity, i.e., it has both magnitude and direction

$$d\phi = \vec{\nabla}\phi \cdot \vec{dl}$$
$$= |\vec{\nabla}\phi||dl| \cos\alpha$$

where α is the angle between $\vec{\nabla}\phi$ & $\vec{dl}$. Clearly the maximum change of ϕ takes place in the direction $\alpha = 0$. It means $d\phi$ is largest when we move in the direction of $\vec{\nabla}\phi$. Or in other words $\vec{\nabla}\phi$ points in the direction of maximum increase of the function ϕ.

The gradient of a scalar function ϕ is a vector whose magnitude is equal to the maximum rate of change of ϕ with respect to the space variables and whose direction is along the change.

Lamellar (non-Curl) vector field

If a vector field $\vec{A}$ can be expressed as the gradient of a scalar field, then $\vec{A}$ is called a lamellar vector field.

A pure electric field $\vec{E}$ can be expressed as gradient of a scalar potential ϕ, i.e.,

$\vec{E} = -\nabla\phi$, thus $\vec{E}$ is a lamellar vector field.

The Divergence of a vector function:

The divergence of a vector field at any point is defined as the amount of flux diverging through the surface enclosing unit volume.

mathematically

$$\text{div}\,\vec{A} = \vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence of a vector function $\vec{A}$ is itself a scalar $\vec{\nabla}\cdot\vec{A}$ (divergence of a scalar quantity is meaningless)

Let there be a vector field $\vec{A}$ in certain region and v be the infinitesimal volume element enclosed by an infinitely small closed surface $\vec{S}$ surrounding a point $P(x, y, z)$.

$$\text{div}\,\vec{A} = \lim_{v\to 0}\frac{\iint_S \vec{A}\cdot d\vec{S}}{v}$$

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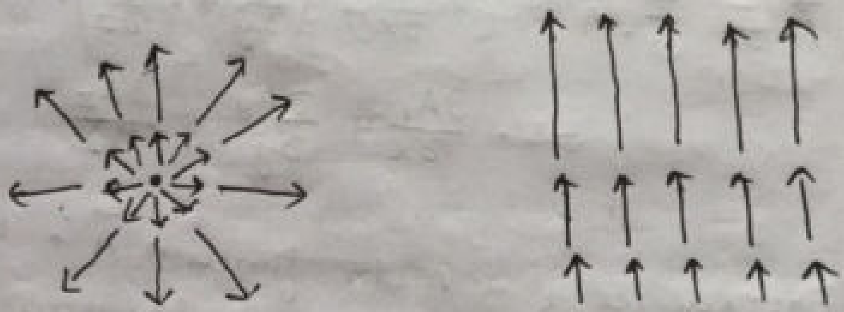
Physical significance

If $\vec{A}$ represents the velocity of a moving fluid at any point P, then div $\vec{A}$ gives the rate at which the fluid is diverging per unit volume from the point P.

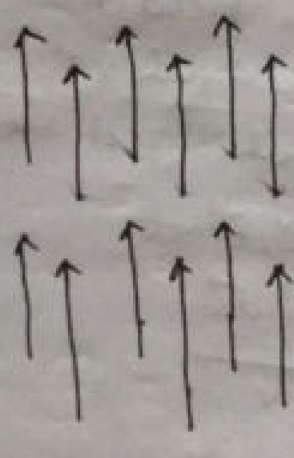
If div $\vec{A} > 0$, at any point P, then either the fluid is expanding or the point P, is a source & if div $\vec{A} < 0$, then either the fluid is contracting or the point P is a sink.

If div $\vec{A} = 0$, then flux of $\vec{A}$ entering any element of space is exactly balanced by the flux leaving it.

A vector $\vec{A}$ that satisfies the condition div $\vec{A} = 0$, is called a solenoidal vector.



Non-Zero divergence.



zero divergence.

Curl of a vector function

The curl of a vector field at a point is defined as the maximum value of the line integral of the vector expressed per unit area surrounding the point and is directed along the normal to the plane of the area.

If $\vec{A}$ be the vector at any point P and ΔS an infinitesimal area surrounding P, then

$$\text{curl}\,\vec{A} = \lim_{\Delta S \to 0} \frac{\left[\oint \vec{A}\cdot d\vec{l}\,\right]_{max}}{\Delta S}\,\hat{n}$$

mathematically

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Curl is a measure of how much the vector $\vec{A}$ curls around the point of question.

If $\nabla \times \vec{A} = 0$, then the vector field $\vec{A}$ is called irrotational. Otherwise $\nabla \times \vec{A} \neq 0$, then $\vec{A}$ is rotational.

Gauss' Divergence theorem

The theorem states that the surface integral of the normal component of a vector taken around a closed surface is equal to the integral of the divergence of the vector taken over the volume enclosed by the surface.

If V be the volume bounded by a closed surface S and $\vec{A}$ is a vector function of position with ~~contin~~ continuous derivatives, then according to the divergence theorem of Gauss'

$$\oiint \vec{A}.\hat{n}\,ds = \iiint_V (\nabla.\vec{A})\,dV$$

where $\hat{n}$ is the outward unit normal to S.

Gauss' divergence theorem thus express the relation between surface and volume integrals.

Stokes' theorem

The line integral of the tangential component of a vector taken around a simple closed curve is equal to the surface integral of the normal component of the curl of the vector taken over any surface having the curve as the boundary.

mathematically, the theorem states that if S is an open two sided surface bounded by a simple closed curve C, and if $\vec{A}$ be any continuously differentiable vector point function, then

$$\oint_C \vec{A}.d\vec{r} = \iint_S (\nabla\times\vec{A}).\hat{n}\,ds = \iint_S (\nabla\times\vec{A}).d\vec{s}$$

where the boundary is traversed in the positive direction. $\hat{n}$ is the outward drawn unit normal to the element of surface ds.

Stokes' theorem thus expresses the relation between line integral and surface integral of a vector.

Gauss' Law

Gauss' law (or Gauss theorem) states that the net outward electric flux through any closed surface in an electric field is equal to $\frac{1}{\epsilon_0}$ times the total charge or zero according to the closed surface encloses the charge or not, where ϵ_0 is the permittivity of the medium.

Thus for a closed surface S enclosing a charge q, the law can be mathematically expressed as

$$\oint_S \vec{E}.d\vec{s} = \frac{1}{\epsilon_0}q, \quad \text{when S encloses q}$$
$$= 0 \quad \text{when S does not enclose q.}$$

Here $\vec{E}$ represents the electric field at the centre of an elementary area ds. The above equation often known as integral form of Gauss' law.

Gauss' Law in Differential form:

From the integral form

$$\oint_E \vec{E}\,\vec{ds} = \frac{q}{\epsilon_0} \quad \text{--- (1)}$$

According to Gauss' diverge theorem

$$\oint_S \vec{E}\cdot\vec{ds} = \int_V (\nabla\cdot\vec{E})\,dV \quad \text{--- (2)}$$

where 'V' represents the volume having surface S as the boundary. For contineous charge distribution

$$q = \int_V \rho\, dV \quad \text{--- --- (3)}$$

where ρ is the volume charge density

$$\therefore \int_V \nabla\cdot\vec{E}\,dV = \frac{1}{\epsilon_0}\int_V \rho\,dV$$

or $$\int_V \left(\nabla\cdot\vec{E} - \frac{\rho}{\epsilon_0}\right) dV = 0$$

or $$\nabla\cdot\vec{E} - \frac{\rho}{\epsilon_0} = 0 \quad \text{or} \quad \nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{--- (4)}$$

Equation (4) is known as the differential form of Gauss' law.

Gauss's law of magnetostatics:

The rate of change of magnetic flux through a closed surface is always zero.

$$\oint \vec{B}\cdot\vec{ds} = 0$$

This also signifies that monopole cannot exist.

Faraday's law of Electromagnetic induction:

Faraday in 1831 observed experimentally that whenever magnetic flux linked with a closed circuit changes, an electromotive force (e.m.f) is induced in the same. The e.m.f so developed is called induced e.m.f and the resulting current in the circuit is known as the induced current. The phenomenon is called electromagnetic induction.

The results of Faraday's experiment led to the development of the following two laws of the electromagnetic induction.

1. **Neumann's law:** The induced e.m.f in a circuit is equal to the time rate of change of the magnetic flux linked with the circuit.

If ϕ be the flux linked with the circuit at any time t, then $\frac{d\phi}{dt}$ gives the time rate of change of flux. According to this law, the magnitude of induced e.m.f

$$|e| = \frac{d\phi}{dt}$$

2. **Lenz's law:** The direction of induced e.m.f or the current is such that it will oppose the very cause for which that is due (i.e. change of flux in the circuit).

$$e = -\frac{d\phi}{dt}$$

The negative sign signifies that e opposes the change of flux. That is why induced e.m.f is sometimes referred as back e.m.f. If R be the resistance of the closed circuit, then induced current

$$i = \frac{e}{R} = -\frac{1}{R}\frac{d\phi}{dt}$$

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Integral form of Faraday's law of electromagnetic induction:

If ϕ be the magnetic flux with the circuit at any time t, then the induced e.m.f is given by

$$e = -\frac{d\phi}{dt} \quad \text{--- (1)}$$

If $\vec{E}$ be the electric field in space, then by definition, the e.m.f around a closed path C is

$$e = \oint_C \vec{E}.\vec{dl}$$

$$\therefore \oint \vec{E}.\vec{dl} = -\frac{d\phi}{dt} \quad \text{--- (2)}$$

If $\vec{B}$ be the magnetic induction vector and $\vec{ds}$ an element of surface, then

$$\phi = \int_S \vec{B}.\vec{ds}$$

$$\therefore \oint \vec{E}.\vec{dl} = -\frac{d}{dt}\int_S \vec{B}.\vec{ds}$$

or

$$\boxed{\oint_C \vec{E}.\vec{dl} = -\int_S \frac{\partial \vec{B}}{\partial t}.\vec{ds}} \quad \text{--- (3)}$$

Differential form of Faraday's law of electromagnetic induction.

By Stoke's law

$$\oint_C \vec{E}.\vec{dl} = \int_S (\nabla \times \vec{E}).\vec{ds}$$

Using eqn (3)

$$\int_S (\nabla \times \vec{E}).\vec{ds} = -\int_S \frac{\partial \vec{B}}{\partial t}.\vec{ds}$$

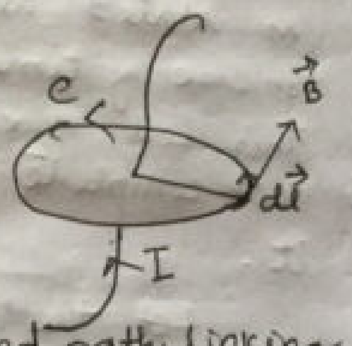
Since S is arbitrary

$$\therefore \boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Displacement Current

Ampere's circuital law in most general form is given by

$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 \iint_s \vec{J}\cdot d\vec{s} \quad \text{--- (1)}$$



where $\vec{B}$ is the magnetic induction due to a current

$$I = \iint_s \vec{J}\cdot d\vec{s} \quad \text{--- (2)}$$

in a conductor and C is closed path linking current I. $\vec{J}$ is the current density and s is the cross-sectional area of the conductor.

By Stokes' law

$$\oint \vec{B}\cdot d\vec{l} = \iint_s (\nabla\times\vec{B})\cdot d\vec{s}$$

$$\therefore \iint_s (\nabla\times\vec{B})\cdot d\vec{s} = \mu_0 \iint \vec{J}\cdot d\vec{s} \quad \text{--- (3)}$$

or

$$\nabla\times\vec{B} = \mu_0\vec{J} \quad \text{--- (4)}$$

Taking divergence on both side

$$\nabla\cdot(\nabla\times\vec{B}) = \mu_0(\nabla\cdot\vec{J})$$

Since $\nabla\cdot(\vec{\nabla}\times\vec{B}) = 0$ & $\mu_0 \neq 0$

$$\therefore \nabla\cdot\vec{J} = 0 \quad \text{--- (5)}$$

Now the equation of continuity is

$$\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \quad \text{--- (6)}$$

when ρ is the charge density

from (5) $\frac{\partial\rho}{\partial t} = 0$ $[\because \nabla\cdot\vec{J} = 0]$

where ρ = constant of time.

This means that Ampere's circuital law is valid on in case where charge density is static, i.e. steady state condition of charge flow, so that $\dot{\rho}=0$. Thus Ampere's circuital law, as stated above, in case where $\frac{\partial \rho}{\partial t} \neq 0$, i.e. in case of time dependent field does not hold good. This led Maxwell to assume that equation ① is not complete and some thing is to be added to it.

Let the quantity to be added to the right hand side of equation ① $\vec{J}_D$, then

$$(\nabla \times \vec{B}) = \mu_0 (\vec{J} + \vec{J}_D) \quad \text{—— ④}$$

Taking the diverge on both sides

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot (\vec{J} + \vec{J}_D)$$

$$0 = \mu_0 (\nabla \cdot \vec{J} + \nabla \cdot \vec{J}_D)$$

or $$\nabla \cdot \vec{J}_D = - \nabla \cdot \vec{J}$$

$$= \frac{\partial \rho}{\partial t} \quad \left[\because \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \right]$$

$$= \frac{\partial}{\partial t} (\nabla \cdot \vec{B}) \quad \left[\because \rho = \nabla \cdot \vec{D} \right]$$

$\vec{D}$ is the electric displacement vector

$$\therefore \quad \nabla \cdot \vec{J}_D = \frac{\partial}{\partial t} (\nabla \cdot \vec{D}) = 0$$

$$\nabla \cdot \left[\vec{J}_D - \frac{\partial \vec{D}}{\partial t} \right] = 0$$

$$\therefore \quad \vec{J}_D = \frac{\partial \vec{D}}{\partial t}$$

Now the modified form of Ampere's circuital law takes the form

$$\nabla \times \vec{B} = \mu_0\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

$\vec{J}_D = \frac{\partial \vec{D}}{\partial t}$ is known as displacement current density which is different from charge transported current density $(\vec{J})$

<u>Important features of displacement current:</u>

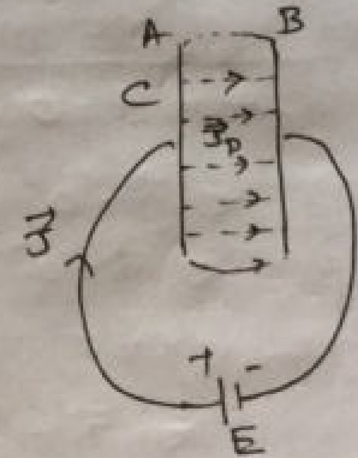
(i) Displacement current is a current only in the sense that it produces a magnetic field. It has none of the other properties of current as it is not related to motion of charges.

(ii) The magnitude of $\vec{J}_D$ is equal to the time rate of change of displacement vector $\vec{D}$. $\vec{J}_D$ may have a certain value in vacuum, where $J = 0$ due to absence of charge.

(iii) Displacement current makes the total current continuous across the discontinuity in conduction current.

(iv) In a conductor, the displacement current is negligible as compared to the conduction current.

The displacement current $\vec{J}_D$ and conduction current $\vec{J}$ are shown in a simple circuit containing a condenser C and an source of e.m.f E.



$$\vec{J}_{total} = \vec{J}_D + \vec{J}$$

Distinction between conduction current and displacement current:

Conduction current	Displacement current
i) It is due to the actual flow of charge in a conductor	i) It is due to the time varying electric field in a dielectric.
ii) It is given by $I_c = \frac{dq}{dt}$	ii) It is given by $I_d = \epsilon_0 A \frac{dE}{dt}$
iii) Conduction current density is given by $\vec{J} = \sigma \vec{E}$, σ being the electrical conductivity & $\vec{E}$ be the electric field	iii) Displacement current density is given by $\vec{J}_D = \frac{\partial \vec{D}}{\partial t} = \epsilon \frac{\partial \vec{E}}{\partial t}$, ϵ being the permittivity of the medium.
iv) It obeys ohm's law	iv) It does not obey ohm's law
v) In a good conductor $J >> J_D$	(v) In a dielectric $J_D >> J$

Maxwell's Electromagnetic equation:

Maxwell's electromagnetic equation are the fundamental equations concerning electromagnetic theory. Following are the four Maxwell's equations in differential form:

1. $\nabla\cdot\vec{D} = \rho$ [Differential form of Gauss's law in electrostatics]

2. $\nabla\cdot\vec{B} = 0$ [Differential form of Gauss's law in magnetostatics]

3. $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ [Differential form of Faraday's law of electromagnetic induction]

4. $\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$ [Maxwell's modified form of Ampere's circuital law]

where $\vec{D} = \epsilon\vec{E}$ electric displacement vector, ϵ being the permittivity of the medium and $\vec{E}$, the electric field intensity

ρ = electric charge density.

$\vec{B} = \mu\vec{H}$, μ is the magnetic ~~permeability~~ permeability & $\vec{H}$, the magnetic field intensity

Maxwell's equations in different media:

(a) Free space (or vacuum)

Here $\rho = 0$ & electrical conductivity $\sigma = 0$,
$\vec{D} = \epsilon_0\vec{E}$ & $\vec{B} = \mu_0\vec{H}$: Here μ_0 & ϵ_0 respectively the permeability & permittivity of free space

Thus

(1) $\nabla\cdot\vec{D}=0$ & $\nabla\cdot\vec{E}=0$

(2) $\vec{\nabla}\cdot\vec{B}=0$ & $\nabla\cdot\vec{H}=0$

(3) $\nabla\times\vec{E}=-\mu_0\frac{\partial\vec{H}}{\partial t}$

(4) $\nabla\times\vec{H}=\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

(b) Conducting medium;

In conducting medium the positive and negative charges are present in equal amount giving $\rho=0$. Also the displacement current density $J_D\simeq0$ so

(1) $\nabla\cdot\vec{D}=0$ $(\vec{D}=\epsilon\vec{E})$

(2) $\nabla\cdot\vec{B}=0$ $(\vec{B}=\mu H)$

(3) $\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$

(4) $\nabla\times\vec{H}=\vec{J}$ $(\vec{J}=\sigma E)$

$\mu=$ permeability, $\epsilon=$ permittivity
$\sigma=$ conductivity

(c) Dielectric medium

Here $\rho=0$, $\sigma=0$ so $\vec{J}=\sigma\vec{E}$

1) $\nabla\cdot\vec{D}=0$

2) $\nabla\cdot\vec{B}=0$

3) $\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$

4) $\nabla\times\vec{B}=\frac{\partial\vec{D}}{\partial t}$

(d) Maxwell's equation for static $\vec{E}$ & $\vec{H}$

Here $\frac{\partial \vec{B}}{\partial t} = 0$ & $\frac{\partial \vec{D}}{\partial t} = 0$

1) $\nabla \cdot \vec{D} = 0$

2) $\nabla \cdot \vec{B} = 0$

3) $\nabla \times \vec{E} = 0$

4) $\nabla \times \vec{H} = \vec{J}$

Maxwell's equation in Integral from

① $\int_s \vec{D} \cdot d\vec{s} = \int_v \rho dv$ or $\oint_s \vec{E} \cdot d\vec{s} = q/\epsilon$

② $\oint_s \vec{B} \cdot d\vec{s} = 0$

③ $\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{s}$

④ $\oint H \cdot dl = \int_s \left(\vec{J} + \frac{\partial D}{\partial t} \right) \cdot d\vec{s}$

Physical significance of Maxwell's equation:

1. $\nabla \cdot \vec{E} = \rho$

(i) The first Maxwell's electromagnetic equation $\nabla \cdot \vec{D} = \rho$ is a steady state equation as it is independent of time.

(ii) The ~~ele~~ equation represents Gauss's law in electrostatics

$$\iint_S \vec{E} \cdot d\vec{s} = \frac{q}{\epsilon} \quad \text{where } q = \iiint_V \rho dv$$

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S is the surface enclosing volume V.

(iii) The equation represents that the electric lines of force are not closed lines ($\because \nabla\cdot\vec{D} \neq 0$). The lines of force originating from the positive charge (source) and terminate in the negative charge (sink)

(iv) If $\vec{D}$ is known, then the scalar, ρ can be determined from the equation $\nabla\cdot\vec{D} = \rho$.

2. $\nabla\cdot\vec{B} = 0$

(i) The equation represents a steady state, independent of time

(ii) This also ~~repre~~ known as differential form of Gauss's law in magnetostatics.

$$\iiint_V (\mathrm{div}.\vec{B}).dv = 0 \ \& \ \oiint \vec{B}.d\vec{s} = 0$$

(iii) $\nabla\cdot\vec{B} = 0$ signifies that the magnetic flux lines are closed curves; i.e. magnetic monopole cannot exist.

3. Maxwell's third equation : $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

(i) This equation is time varying and represents Faraday's law of electromagnetic induction.

(ii) The negative sign in the right-hand side of the above equation signifies the truth of Lenz's law.

4. $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

(i) This is a time dependent equation.

(ii) This is a modified form of Ampere's circuital law and states that, a changing electric field produces a changing magnetic field. Also according to Faraday's law a changing magnetic field produces a changing electric field.

(iii) Alternate production of electric & magnetic field leads to the propagation of electromagnetic waves in a medium.

Electromagnetic wave in free space

Maxwell's electromagnetic equation in free space (i.e. $\rho = 0, \sigma = 0, \mu = \mu_0, \epsilon = \epsilon_0$) are as follows

1. $\nabla \cdot \vec{E} = 0$ - (1)
2. $\nabla \cdot \vec{H} = 0$ - (2) $[\because B = \mu_0 H]$
3. $\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$ - (3)
4. $\nabla \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ - (4)

Now taking curl on both sides of equation (3)

$$\nabla \times (\nabla \times \vec{E}) = -\mu_0 \frac{\partial}{\partial t}(\nabla \times \vec{H})$$

Using eq^n (4) on the right hand side.

$$\nabla \times (\nabla \times \vec{E}) = -\mu_0 \frac{\partial}{\partial t}\left(\epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$= -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

or $\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$

$[\because \nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}]$

For free space $\nabla\cdot\vec{E}=0$

or $\nabla^2\vec{E} - \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = 0$

or $\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$ $\left[\mu_0\epsilon_0 = \frac{1}{c^2}\right]$ - (5)

Taking curl on both sides of equation (4)

$\nabla\times\nabla\times\vec{H} = \epsilon_0\frac{\partial}{\partial t}(\nabla\times\vec{E})$

or $\nabla(\nabla\cdot\vec{H}) - \nabla^2\vec{H} = \epsilon_0\frac{\partial}{\partial t}\left(-\mu_0\frac{\partial H}{\partial t}\right)$

$\nabla^2\vec{H} - \mu_0\epsilon_0\frac{\partial^2\vec{H}}{\partial t^2} = 0$ $[\because \nabla\cdot\vec{H}=0]$

$\nabla^2\vec{H} - \frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2}$ --- (6)

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = \frac{1}{\sqrt{8.854\times10^{-12}\,\text{farad/m}\times 4\pi\times10^{-7}}}$$

again $\frac{\mu_0}{4\pi} = 10^{-7}$ web/A m, $\frac{1}{4\pi\epsilon_0} = 9\times10^{9}$ m/farad.

or $c = 3\times10^{8}$ m/sec.

Equations (5) & (6) represents three dimensional wave equation without damping travelling with velocity c. The $\vec{E}$ & $\vec{H}$ vectors progate in free space with velocity of c.

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Solution of plane electromagnetic wave: transverse nature of electromagnetic wave

A plane wave is that whose amplitude of vibration is the same at any point in a plane perpendicular to the specified direction. From the previous discussion, the propagation for electromagnetic waves can be written as

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \quad \text{(for electric field)}$$

$$\nabla^2\vec{H} - \frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0 \quad \text{(for magnetic field)}$$

Solutions of these plane electromagnetic wave equations are as follows: $[i = \sqrt{-1}]$

$$\vec{E}(r,t) = \vec{E_0}\, e^{i(\vec{K}\cdot\vec{r} - \omega t)} \quad \text{--- (1)}$$

$$\vec{H}(r,t) = \vec{H_0}\, e^{i(\vec{K}\cdot\vec{r} - \omega t)} \quad \text{--- (2)}$$

where $\vec{E_0}$ and $\vec{H_0}$ are the complex amplitudes which are constant in space and time and K the propagation vector given by

$$\vec{K} = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi\nu}{c}\hat{n} = \frac{\omega}{c}\hat{n}$$

$\hat{n}$ being the unit vector in the direction of propagation of wave

$$\nabla\cdot\vec{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) E_0\, e^{i(\vec{K}\cdot\vec{r} - \omega t)}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\, e^{i(K_x x + K_y y + K_z z) - i\omega t}\right]$$

$$\left[\because\ \vec{K}\cdot\vec{r} = (\hat{i}K_x + \hat{j}K_y + \hat{k}K_z)\cdot(\hat{i}x + \hat{j}y + \hat{k}z)\right]$$

$$\nabla\cdot\vec{E} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(\hat{i}E_{ox}+\hat{j}E_{oy}+\hat{k}E_{oz})\,e^{i[(K_x x+K_y y+K_z z)-\omega t]}\right]$$

$$= (E_{ox}\,iK_x + E_{oy}\,iK_y + E_{oz}\,iK_z)\,e^{i(\vec{K}\cdot\vec{r}-\omega t)}$$

$$= i(\hat{i}K_x+\hat{j}K_y+\hat{k}K_z)\cdot(\hat{i}E_{ox}+\hat{j}E_{oy}+\hat{k}E_{oz})\,e^{i(\vec{K}\cdot\vec{r}-\omega t)}$$

$$= i\vec{K}\cdot\vec{E}_0\,e^{i(\vec{K}\cdot\vec{r}-\omega t)} \quad --- (3)$$

or $\nabla\cdot\vec{E} = i\vec{K}\cdot\vec{E}$

similarly $\nabla\cdot\vec{H} = i\vec{K}\cdot\vec{H}$ $\quad$ (4)

Equation (4) is similar to a eigen value equation and it is clear that ∇ is equivalent to $i\vec{K}$

$$\frac{\partial\vec{E}}{\partial t} = \frac{\partial}{\partial t}\left[\vec{E}_0\,e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right] = -i\omega E_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)}$$

or $\dfrac{\partial\vec{E}}{\partial t} = -i\omega\vec{E}$

which suggests $\frac{\partial}{\partial t}$ is equivalent to $(-i\omega)$

Thus Maxwell's equation in free space in terms of the operators (iK) & $(-i\omega)$

$$\vec{K}\cdot\vec{E}=0 \quad --(a)$$
$$\vec{K}\cdot\vec{H}=0 \quad -(b)$$
$$\vec{K}\times\vec{E} = \omega\mu_0\vec{H} \quad -(c)$$
$$-\vec{K}\times\vec{H} = \omega\epsilon_0\vec{E} \quad -(d)$$

From (a) & (b) we note that $\vec{E}$ is perpendicular to $\vec{K}$, and $\vec{H}$ and also $\vec{H}$ is perpendicular to $\vec{K}$. Hence both $\vec{E}$ & $\vec{H}$ are perpendicular to $\vec{K}$. In other words electromagnetic waves are transverse in nature.

Poynting Theorem and Poynting vector

The theorem states that "the rate at which electromagnetic energy in a finite volume decreases with time is equal to the rate of dissipation of energy in the form Joule heat plus the rate at which energy flows out of the volume."

From the Maxwell's third and fourth electromagnetic equations,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{—} \quad (1)$$

&

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{—} \quad (2)$$

Now

$$-\vec{H}\cdot(\nabla \times \vec{E}) = \vec{H}\cdot\frac{\partial \vec{B}}{\partial t} \quad \text{—} \quad (3)$$

and

$$\vec{E}\cdot(\nabla \times \vec{H}) = \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t} \quad \text{—} \quad (4)$$

Adding (3) & (4) we get

$$-\vec{H}\cdot(\nabla \times \vec{E}) + \vec{E}\cdot(\nabla \times \vec{H}) = \vec{E}\cdot\vec{J} + \left[\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t}\right]$$

$$-\left[\vec{H}\cdot(\nabla \times \vec{E}) - \vec{E}\cdot(\nabla \times \vec{H})\right] = \vec{E}\cdot\vec{J} + \left[\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t}\right]$$

$$-\nabla\cdot(\vec{E}\times\vec{H}) = \vec{E}\cdot\vec{J} + \left[\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t}\right] \quad \text{—} \quad (5)$$

$$\left[\because \vec{B}\cdot(\nabla \times \vec{A}) - \vec{A}\cdot(\nabla \times \vec{B}) = \nabla\cdot(\vec{A}\times\vec{B})\right]$$

Now

$$\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} = \mu \vec{H}\cdot\frac{\partial \vec{H}}{\partial t} = \frac{1}{2}\mu\frac{\partial}{\partial t}(\vec{H}\cdot\vec{H}) \quad [\because \vec{B} = \mu\vec{H}$$

$$= \frac{1}{2}\frac{\partial}{\partial t}(\vec{H}\cdot\vec{B}) \quad \text{—} \quad (6)$$

&

$$\vec{E}\cdot\frac{\partial \vec{D}}{\partial t} = \epsilon\vec{E}\cdot\frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\epsilon\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E}) \quad [\because \vec{D} = \epsilon\vec{E}$$

$$= \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D}) \quad \text{—} \quad (7)$$

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Now equation ⑤ becomes

$$\nabla\cdot(\vec{E}\times\vec{H}) = -\vec{E}\cdot\vec{J} - \frac{\partial}{\partial t}\frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})$$

Integrating over a volume V, then

$$\iiint_V \nabla\cdot(\vec{E}\times\vec{H})\,dv = -\iiint_V (\vec{E}\cdot\vec{J})\,dv - \frac{\partial}{\partial t}\iiint_V \frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})\,dv$$

If S be the surface enclosing volume V, then

$$-\frac{\partial}{\partial t}\iiint_V \frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})\,dv = \iiint_V (\vec{J}\cdot\vec{E})\,dv + \oint_S (\vec{E}\times\vec{H})\cdot d\vec{s} \quad -⑥$$

(we have used Gauss divergence theorem)

Now the term

$$-\frac{\partial}{\partial t}\iiint_V \frac{1}{2}\left[(\vec{E}\cdot\vec{D})+(\vec{H}\cdot\vec{B})\right]dv$$

$$= -\frac{\partial}{\partial t}\left[\iiint_V \frac{1}{2}(\vec{E}\cdot\vec{D})\,dv + \iiint_V \frac{1}{2}(\vec{H}\cdot\vec{B})\,dv\right]$$

$$= -\frac{\partial}{\partial t}\left[U_e + U_m\right]$$

where

$$U_e = \iiint \frac{1}{2}(\vec{E}\cdot\vec{D})\,dv$$

&

$$U_m = \iiint_V \frac{1}{2}(\vec{H}\cdot\vec{B})\,dv$$

are the electric energy and magnetic energy stored in volume V

The left hand side of equation ⑥ represents the time rate of decrease of electromagnetic energy in volume V.

The term $\iiint_V (\vec{J}\cdot\vec{E})\,dv$ represents the rate of dissipation of energy in the form of Joule heat in V.

The term

$\oiint (\vec{E}\times\vec{H})\cdot d\vec{S}$ represents the time rate of ~~change~~ flow of energy from v through the surface S.

$$\vec{E}\times\vec{H} = \vec{S}$$ (do not confuse with surface $\vec{S}$)

So $\vec{S} = \vec{E}\times\vec{H}$ is the energy flowing through unit area and unit time and is known as the Poynting vector.

Hence the Poynting vector ($\vec{S}$) may be defined as the amount of electromagnetic field energy flowing through unit area of the surface in a direction perpendicular to the plane containing $\vec{E}$ & $\vec{H}$ per unit time.