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Strain Energy

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Introduction

- In mechanics, Energy is defined as the capacity to do work, and work is the product of the force and the distance it moves along its direction.
- In solid deformable bodies, the stresses multiplied by the respective areas are the forces and the deformation are the distances.
- The product of the force and deformations is the internal work done in a body by externally applied forces.
- The internal work done is stored in the body as the internal elastic energy of deformation or the elastic strain energy.

Conservation of energy, work and strain

- Conservation of energy is one of the basic law of physics and in a closed system consisting of a structure and the applied force must obeys this law.

  $W = E_s + E_l$

  W = Work Performed

  E_s = Energy stored in the body

  E_l = Energy loss

- Now in a structure, work is performed by the external load moving through a distance and the energy is stored due to elastic deformation of the members.
- If the structure is static there is no kinetic energy in the system with no energy loss due to heat, permanent set etc. The equation reduces to

$W = E_s$

E_s = Elastic strain energy also denoted by "U"

Hence for a conservational structural system

$W = U$

Strain energy/unit volume = u = $1/2 \times \sigma \times \varepsilon$

Total Strain energy = U = $\frac{1}{2} \int \sigma \times \varepsilon \times dv$

where, σ = stress, ε = strain

Real work and Complimentary work

- Work = Force × Displacement
- The work done as the force F moves through a distance $d\Delta$

  $\Delta W = F \times d\Delta$

  Total work done = $W = \int F \times d\Delta$
- If force "F" is three dimensional with components F_x, F_y and F_z

  Total work done $W = \int F_x \times d\Delta_x + \int F_y \times d\Delta_y + \int F_z \times d\Delta_z$

  This work is known as Real work as shown in Fig. 1.

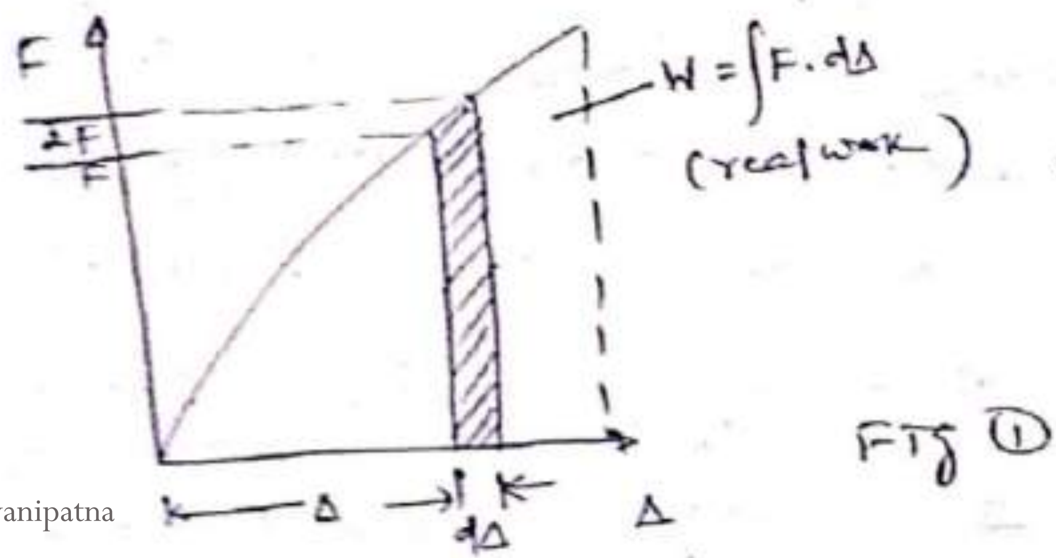


FIG ①

Complimentary work:

$\Delta W_c = \Delta \times dF$

Total Complimentary work, $W_c = \int \Delta \times dF$ as in Fig. 2.

- It is the area above the load deflection curve.
- In linear elastic analysis, load – deflection curve is linear as shown in Fig. 3.

Real work = Complimentary work

$W = W_c = ½ F \times \Delta$

Area below the graph = Area above the graph

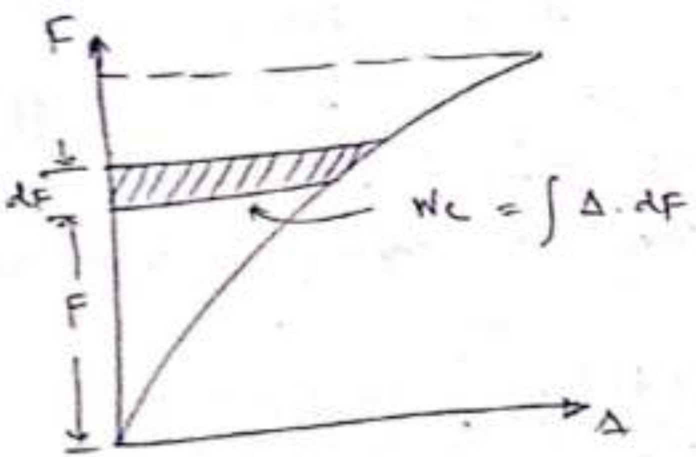


Fig 2

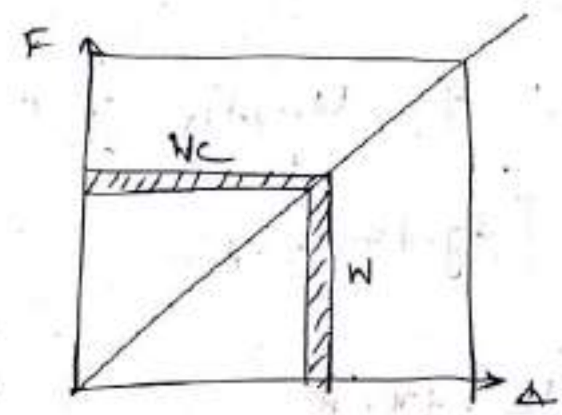
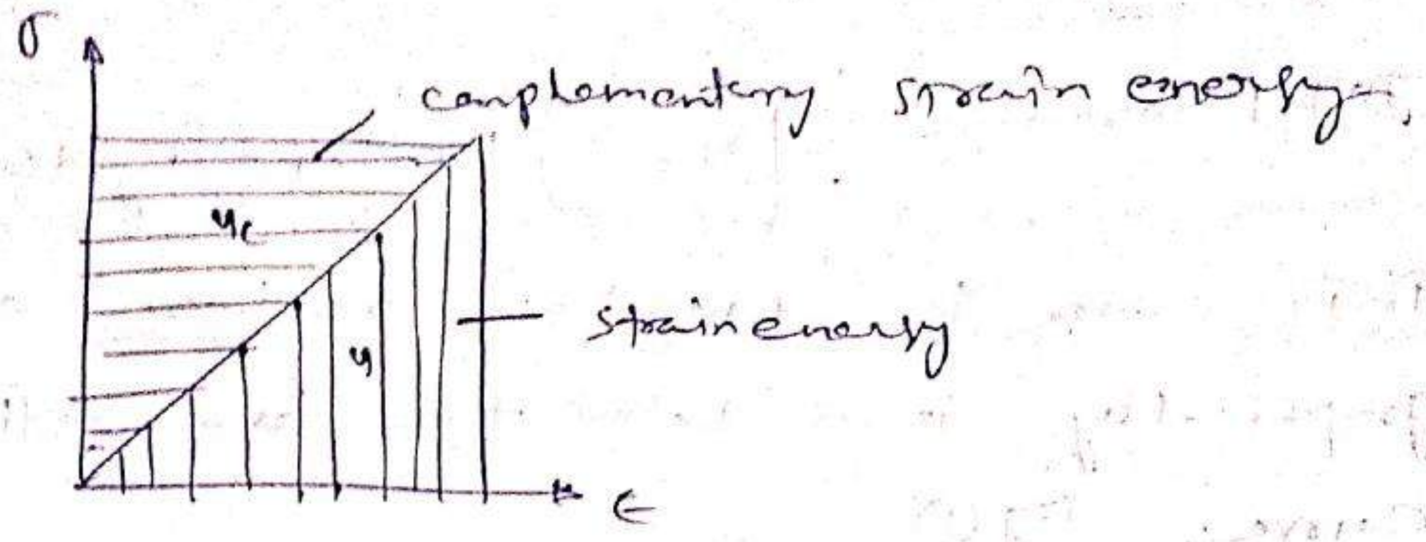


Fig 3

Expression of strain energy for linear elastic members

- Axial loaded members
- Members under Bending moment
- Members under torsional moment on a circular cross section
- Members under shear force on a rectangular section

- Axial loaded members:



$$u = \frac{1}{2}\sigma.\epsilon$$

u = Strain energy per unit volume

$$U = \frac{1}{2}\int \sigma.\epsilon.dv$$

U = Total strain energy.

$$U = \frac{1}{2}\int \sigma.\frac{\sigma}{E}.dx.dy.dz$$

F = External force or load, A = Area of a bar, L = Length of a bar
E = Modulus of elasticity

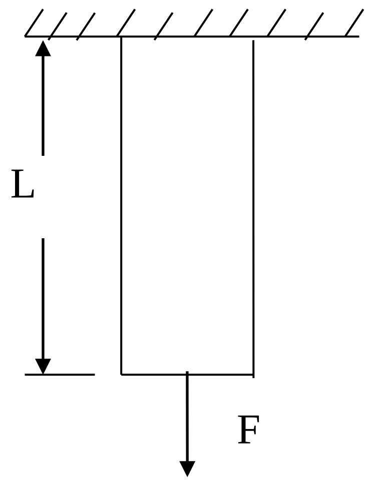
For a member of length 'L',

$$\sigma = \frac{F}{A} \; ; \quad \int dy \cdot dz = A.$$

$$U = \frac{1}{2}\int \frac{F}{\cancel{A}} \cdot \frac{F}{AE} \cdot dx \cdot \cancel{A}$$

$$U = \frac{1}{2}\int \frac{F^2\,dx}{AE}$$

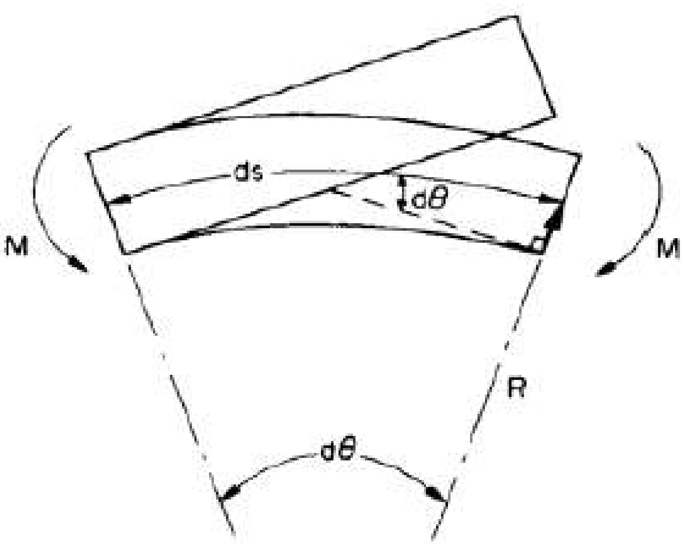
$$\boxed{U = \frac{1}{2}\frac{F^2 L}{AE}} \qquad \boxed{\int dx = L}$$



• Members under Bending moment:

From the pure bending, we know **M/I = σ/y = E/R**
where. M = Bending moment, I = moment of inertia, σ = Bending stress, y = most distant point from the neutral axis, E = modulus of elasticity, R = Radius of curvature

Strain energy = work done = $\frac{1}{2}$ moment x angle turned through (in radians)



$$\sigma = \frac{M}{I} \cdot y$$

$$U = \frac{1}{2} \int \frac{M}{I} \cdot y \cdot \frac{M}{I} \cdot \frac{y}{E} \cdot dx \cdot dy \cdot dz$$

$$= \frac{1}{2E} \int \frac{M^2 \cdot dx}{I^2} \int y^2 \cdot dy \cdot dz$$

$$= \frac{1}{2E} \int \frac{M^2 \cdot dx}{I^2} \cdot I$$

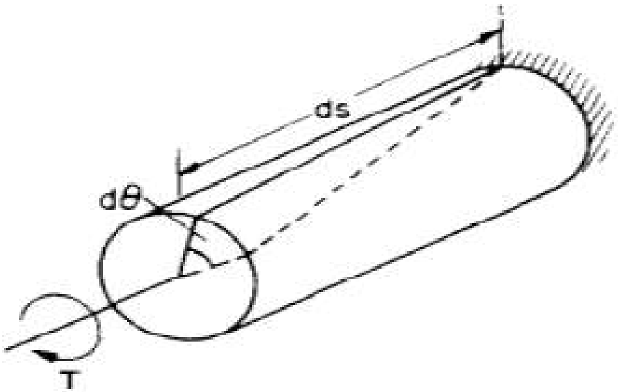
$$= \frac{1}{2E} \int \frac{M^2}{I} dx$$

$$\boxed{U = \frac{1}{2EI} \int M^2 \cdot dx}$$

- Members under torsional moment on a circular cross section:

$$\text{Strain energy} = \text{work done} = \frac{1}{2} T\, d\Theta$$

$$U = \tfrac{1}{2} \int \text{shear stress} \times \text{shear strain} \times \text{volume}$$



$$\tau_{xy} = \frac{T \cdot r}{J}$$

$$\gamma_{xy} \cdot G = \frac{T \cdot r}{J}$$

$$\gamma_{xy} = \frac{T \cdot r}{GJ}$$

G = Rigidity modulus

τ_{xy} = Shear stress

γ_{xy} = Shearing strain

$$\boxed{G = \frac{\tau_{xy}}{\gamma_{xy}}}$$

$$U = \frac{1}{2} \int \tau_{xy} \cdot \gamma_{xy} \cdot dx \cdot dy \cdot dz$$

$$= \frac{1}{2} \int \frac{T \cdot r}{J} \cdot \frac{Tr}{GJ}\, dx \cdot dy \cdot dz$$

T = Torsional moment

r = Radius of shaft

J = Polar moment of Inertia.

$$= \frac{1}{2G}\int \frac{T^2}{J^2}\,dx \int r^2 \cdot dy\,dz$$

$$= \frac{1}{2G}\int \frac{T^2}{J^2}\,dx \cdot J$$

$$\boxed{U = \frac{1}{2GJ}\int T^2\,dx}$$

- Members under shear force on a rectangular section:

V = shear force, I = moment of inertia, b = width of the section, G = shear modulus

$$\tau_{xy} = \frac{VQ}{Ib}, \quad \gamma_{xy} = \frac{VQ}{Ib \cdot G}$$

$$U = \frac{1}{2}\int \frac{VQ}{I \cdot b} \cdot \frac{VQ}{Ib \cdot G} \cdot dx \cdot dy \cdot dz$$

$$= \frac{1}{2G}\int \frac{V^2 Q^2}{I^2 b^2} \cdot dx \cdot dy \cdot dz$$

$$= \frac{1}{2G}\int V^2 \cdot dx \int \frac{Q^2 \cdot dy \cdot dz}{I^2 b^2}$$

Now $\displaystyle\int \frac{Q^2 \cdot dy \cdot dz}{I^2 \cdot b^2} = \frac{1}{I^2}\int \left(\frac{Q}{b}\right)^2 \cdot dA = \frac{1}{A_s}$

A_s is called effective shear area.

$$\boxed{U = \frac{1}{2G} \cdot \int \frac{V^2 \cdot dx}{A_s}}$$

Deflection by Strain Energy Method

- This is also known as real work methods since work done by actual loads are considered.
- From the law of conservation of energy,

  Strain energy = Real work done by loads

$$U = \sum_{0}^{n} \frac{1}{2} \cdot P \Delta$$

This method is used for finding deflection in structure only under the following situations:

- The structure is subjected to a single concentrated load.
- Deflection required is at the loaded point and is in the direction of load.

Thanks

Deflection by Strain Energy Method

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Deflection by Strain Energy Method

- This Method is also called 'Real Work Method'.
- Since, work done by the actual loads are considered.
- From the law of conservation of energy,

Strain Energy (U) = Real work done by loads

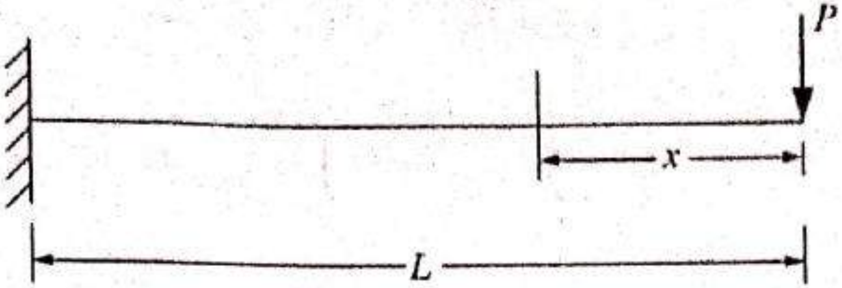
$$U = \sum_{0}^{n} \frac{1}{2} P\Delta$$

- This equation can be used to find out the deflection in beams and frames subjected to bending stresses.

Strain energy method can be used for finding deflection under the following situations:

- The structure is subjected to a concentrated load.
- Deflection required is at the loaded point and is in the direction of load.

Q1. Using strain energy method determine the deflection of the free end of a cantilever of length 'L' subjected to a concentrated load 'P' at the free end.



Solution The bending moment at a distance x from the free end is,

$$M = Px$$

$\therefore$ Strain Energy (S.E.) $= \int_0^L \frac{M^2}{2EI} dx$

$$= \int_0^L \frac{P^2 x^2}{2EI} dx$$

$$= \frac{P^2}{2EI}\left[\frac{x^3}{3}\right]_0^L$$

$$= \frac{P^2 L^3}{6EI}$$

Work done by the load $= \frac{1}{2}P\Delta$, where Δ is the deflection at the free end.

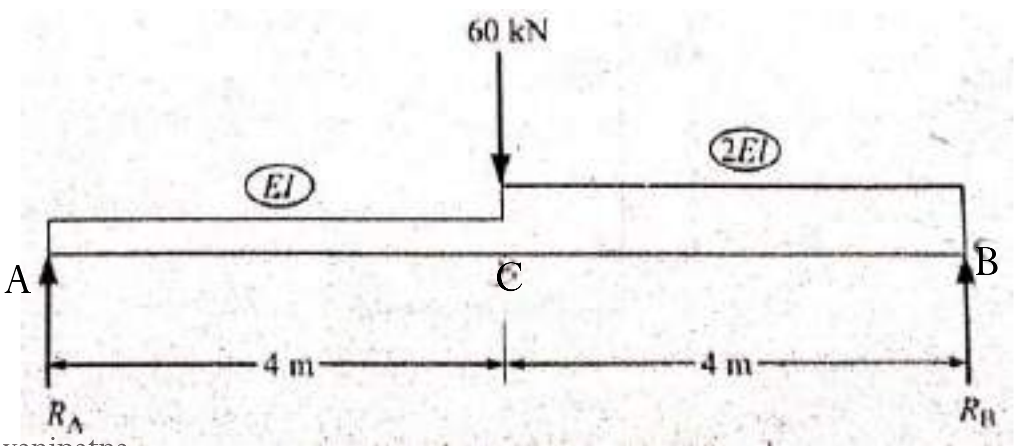
Therefore, from conservation of energy,

$$\text{S.E.} = \text{Work done by external loads}$$

$$\frac{P^2 L^3}{6EI} = \frac{1}{2}P\Delta$$

$$\Delta = \frac{PL^3}{3EI}$$

Q2. Using strain energy method determine the deflection under 60 kN load in the beam shown in Figure.



Solution Reaction $R_A = R_B = 30$ kN

Therefore, bending moment at any distance x from A or at a distance x from B

$$= 30x \text{ kN}$$

$$\therefore \qquad \text{S.E.} = \int_0^4 \frac{(30x)^2}{2EI} dx + \int_0^4 \frac{(30x)^2}{2\times 2EI} dx$$

$$U = \frac{3}{4} \times \frac{900}{EI} \int_0^4 x^2 \, dx$$

$$U = \frac{3}{4} \times \frac{900}{EI} \left[\frac{x^3}{3} \right]_0^4 = \frac{3}{4} \times \frac{900}{EI} \times \frac{4^3}{3}$$

$$U = \frac{14400}{EI}$$

Work done by the load:

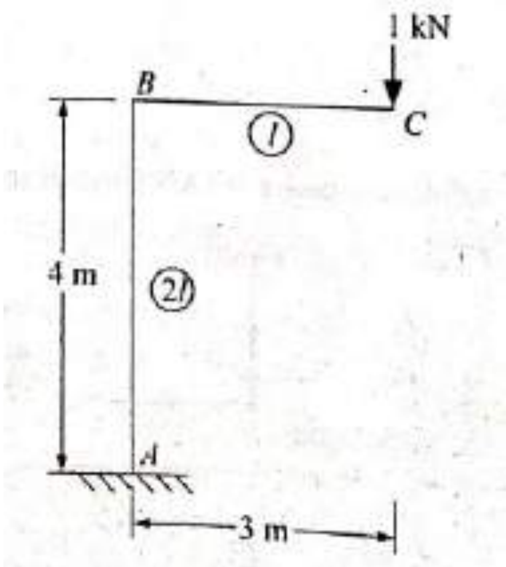
$$W_E = \frac{1}{2} \times P\Delta = \frac{1}{2} \times 60 \times \Delta$$

Equating strain energy of the beam to the work done by load; we get,

$$\frac{14400}{EI} = \frac{1}{2} \times 60 \times \Delta$$

$$\Delta = \frac{480}{EI}$$

Q3. Using strain energy method determine the vertical deflection of point '*C*' in the frame shown in Figure. $E = 200$ kN/mm^2 and $I = 30 \times 10^6$ mm^4.



- The details of bending moment expressions for various portion of the structure is calculated individually for member BC than for member AB, and given data in Tabular form:

Portion	Origin	Limit	Expression
BC	*C*	0 – 3	$1\,x = x$
AB	*B*	0 – 4	3

$$\text{S.E} = \int_0^3 \frac{(x)^2}{2EI}dx + \int_0^4 \frac{(3)^2}{2E\times 2I}dx$$

$$= \frac{1}{2EI}\left[\frac{x^3}{3}\right]_0^3 + \frac{1}{4EI}[9x]_0^4$$

$$= \frac{1}{6EI}\times 3^3 + \frac{1}{4EI}\times 9\times 4$$

$$= \frac{13.5}{EI}$$

Note: As the bending moment is given in kN and metres, *EI* should be used as kNm^2.
i.e. 1 $kNmm^2 = 1\times 10^{-6}$ kNm^2

$$\text{Work done} = \frac{1}{2}\times 1\times \Delta = \frac{\Delta}{2}$$

Equating work done to strain energy, we get

$$\frac{\Delta}{2} = \frac{13.5}{EI}$$

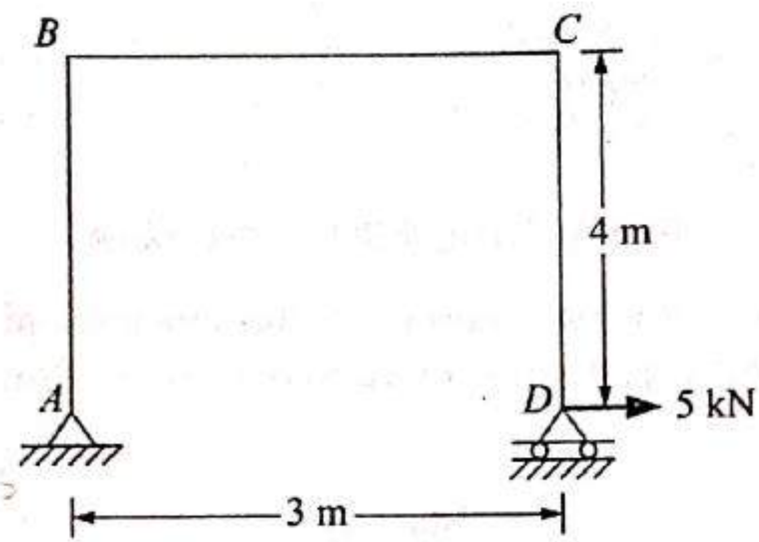
$$\Delta = \frac{27}{EI}$$

$$EI = 200 \times 30 \times 10^6 \times 10^{-6} = 6000 \text{ kNm}^2$$

$$\Delta = \frac{27}{6000} \text{ m}$$

$$= 0.045 \text{ m} = 4.5 \text{ mm}$$

Q4. Using strain energy method determine the horizontal deflection of the roller end 'D' of the portal frame shown in Figure. EI = 8000 kNm^2 throughout.



- The details of bending moment expressions for various portion of the structure is calculated individually for member CD, BC than for member AB, and given data in Tabular form:

Portion	*CD*	*BC*	*AB*
Origin	*D*	*C*	*B*
Limit	0 – 4	0 – 3	0 – 4
Mx	$5x$	20	$20 - 5x$

$$S.E = \int_0^4 \frac{(5x)^2}{2EI}dx + \int_0^3 \frac{(20)^2}{2EI}dx + \int_0^4 \frac{(20-5x)^2}{2EI}dx$$

$$= \frac{1}{2EI}\left[\frac{25x^3}{3}\right]_0^4 + \frac{1}{2EI}[400x]_0^3 + \frac{1}{2EI}\left[400x - 200\frac{x^2}{2} + \frac{25x^3}{3}\right]_0^4$$

$$= \frac{266.67}{EI} + \frac{600}{EI} + \frac{1}{2EI}\left[1600 - 1600 + \frac{25\times 64}{3}\right]$$

$$= \frac{1133.33}{EI}$$

Work done $= \frac{1}{2}\times P\times\Delta = \frac{1}{2}\times 5\Delta = 2.5\Delta$

Equating S.E. to work done, we get, $2.5\Delta = \frac{1133.33}{EI}$

$$\Delta = \frac{453.33}{EI} = \frac{453.33}{8000} = 0.0567\,\text{m}$$

$$= 56.7\,\text{mm}$$

Thanks

Law of Maxwell's Reciprocal Theorem

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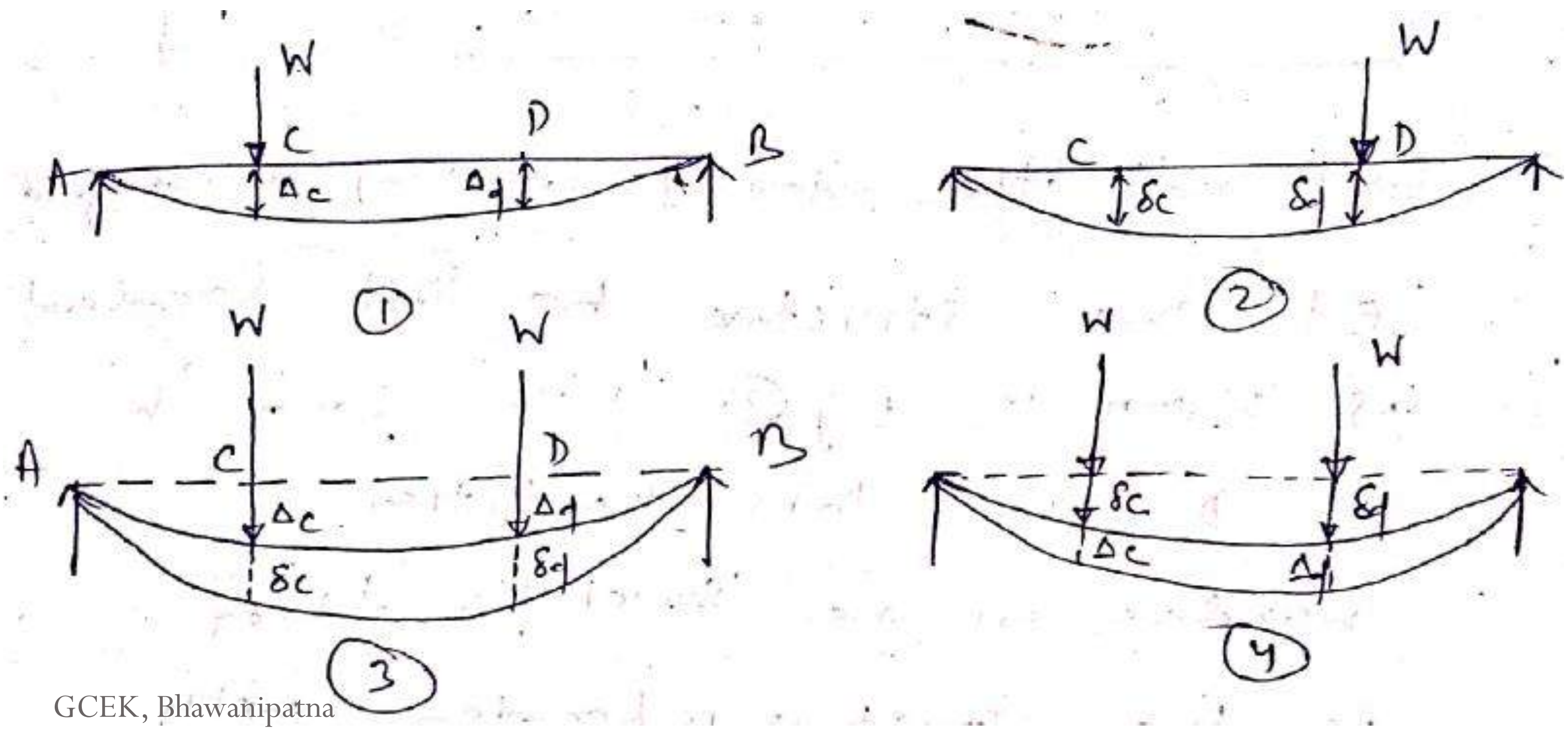
Law of Reciprocal deflections or Maxwell's Reciprocal Theorem:

- In any beam or truss the deflection at any point 'D' due to a load 'W' at any other point 'C' is the same as the deflection at 'C' due to the same load 'W' applied at 'D'.



- Figure 1 shows a structure AB carrying a load 'W' applied at any other point 'C'.

  Let the deflection at 'C' = Δ_c

  Let the deflection at any other point = Δ_d

- Figure 2 shows a same structure AB carrying the same load 'W' at 'D'.

  Let the deflection at 'C' = δ_c

  Let the deflection at 'D' = δ_d

- Let the structure loaded as shown in Figure 1.

  Work done on the structure = ½ W Δ_c

- As the structure is loaded with a load 'W' at 'C', let another equal load 'W' be applied at 'D' . There will be further deflection δ_c and δ_d at 'C' and 'D' as shown in Figure 3.

Total work done at this stage = ½ W Δ_c + ½ W δ_d + W δ_c---------1

- Let now the order of loading be changed. Let the structure be first loaded as shown in Figure 2 with load ‘W’ at ‘D’, for this condition

Work done on the structure = ½ W δ_d

- As the structure is loaded with load ‘W’ at ‘D’, let an equal load ‘W’ be applied at ‘C’ . Further deflections Δ_c and Δ_d occurs at ‘C’ and ‘D’ respectively as shown in Figure 4.

Total work done at this stage = ½ W δ_d + ½ W Δ_c + W Δ_d ---------2

- Equating the two expressions 1 and 2 obtained for the total work done when both loads are present on the structure

$$\tfrac{1}{2} W \Delta_c + \tfrac{1}{2} W \delta_d + W \delta_c = \tfrac{1}{2} W \delta_d + \tfrac{1}{2} W \Delta_c + W \Delta_d$$

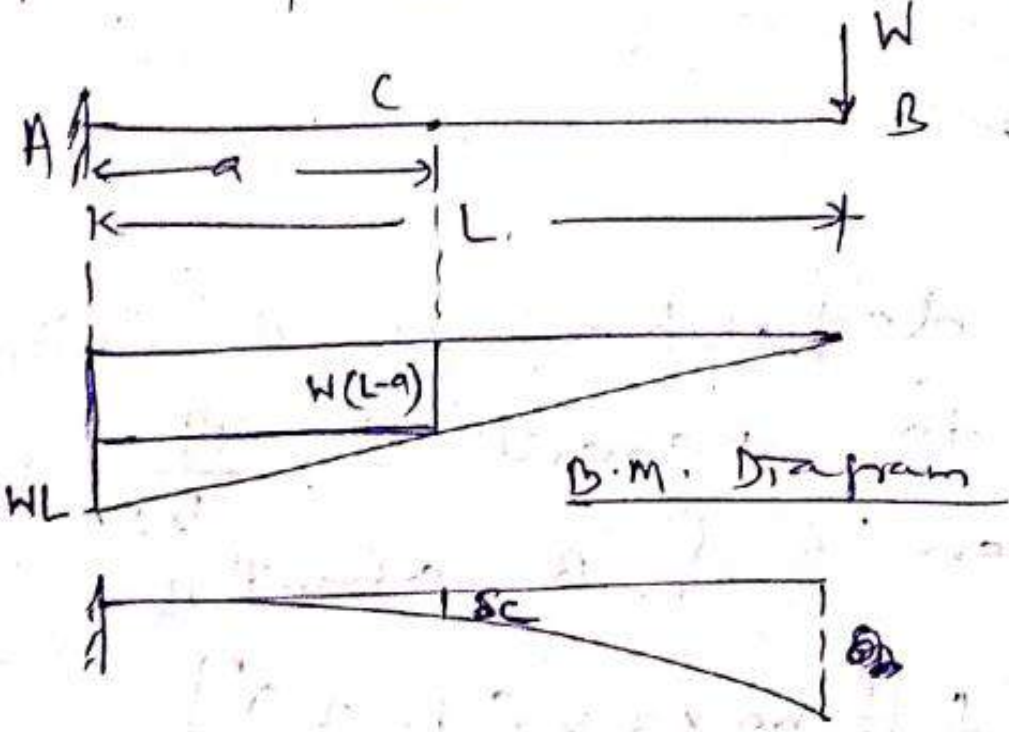
$$W \delta_c = W \Delta_d$$

$$\boldsymbol{\delta_c = \Delta_d}$$

- The deflection at ‘C’ due to load ‘W’ at ‘D’ = Deflection at ‘D’ due to load ‘W’ at ‘C’

Proof of Maxwell's Reciprocal theorem

Consider a Cantilever AB loaded with load 'W' at free end 'B'.



The deflection at 'C', distance 'a' from 'A' 'B' equal to moment of $\frac{M}{EI}$ diagram between 'A' & 'C' about 'C'..

$$\delta_C = \frac{1}{EI}\left[W(l-a).a \times \frac{a}{2} + \frac{1}{2} \times Wa \times a \times \frac{2}{3}a \right]$$

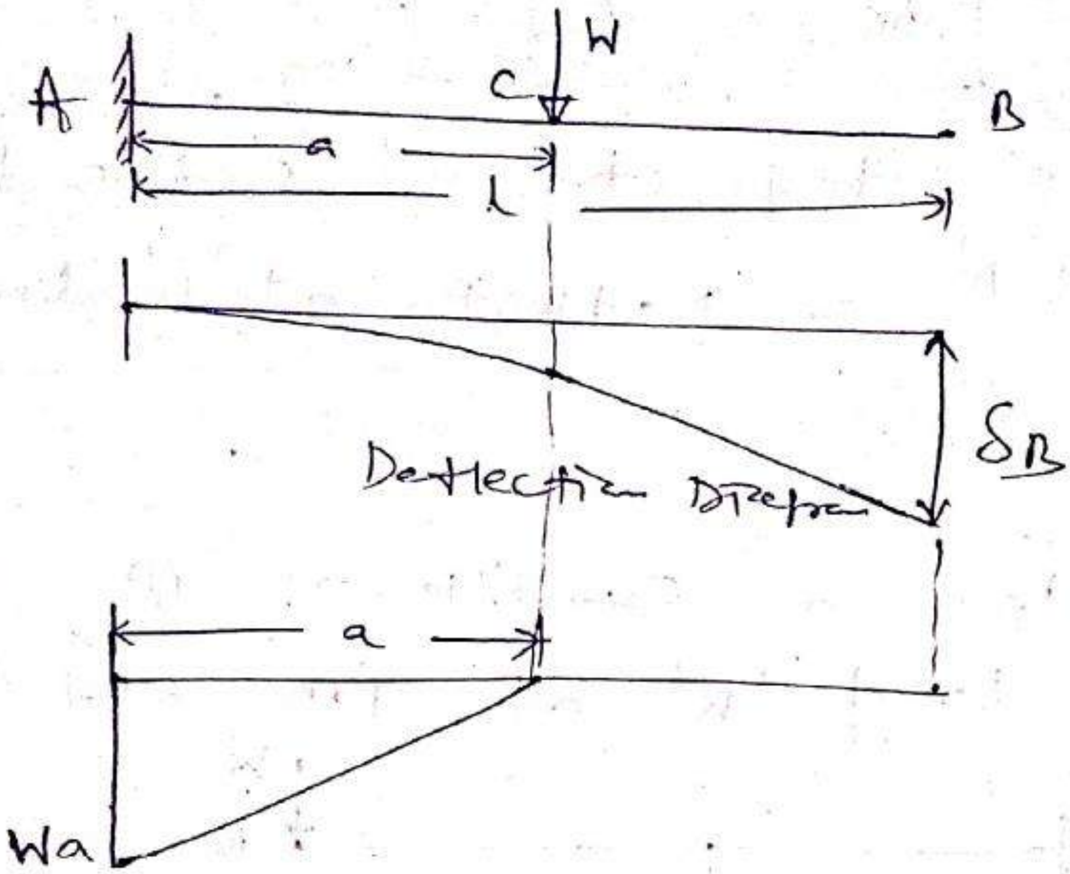
$$= \frac{1}{EI}\left[W.(l-a)\frac{a^2}{2} + \frac{Wa^3}{3} \right]$$

$$= \frac{1}{EI} Wa^2 \left[\frac{l-a}{2} + \frac{a}{3} \right]$$

$$= \frac{Wa^2}{EI}\left[\frac{3l-3a+2a}{6} \right]$$

$$\delta_C = \frac{Wa^2}{EI}\left[\frac{3l-a}{6} \right] = \frac{Wa^2}{2EI}\left(l - \frac{a}{3} \right)$$

Now consider the same cantilever loaded with load 'W' at 'c'.



The deflection at 'B' will be equal to moment of $\frac{M}{EI}$ diagram between 'A' & 'B' about 'B'.

$$\delta_B = \frac{1}{EI}\left[\frac{1}{2} Wa \times a \times \left(l - \frac{a}{3}\right)\right]$$

$$\boxed{\delta_B = \frac{Wa^2}{2EI}\left(l - \frac{a}{3}\right)}$$

Thus $\delta_C = \delta_B$

Thanks

Betti's Theorem

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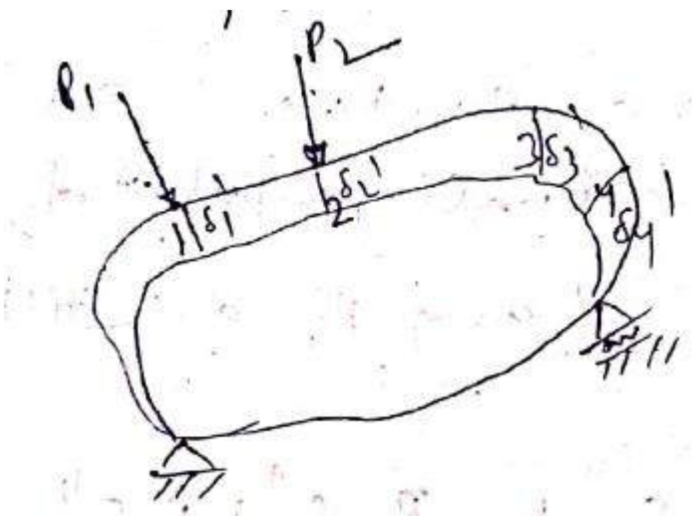
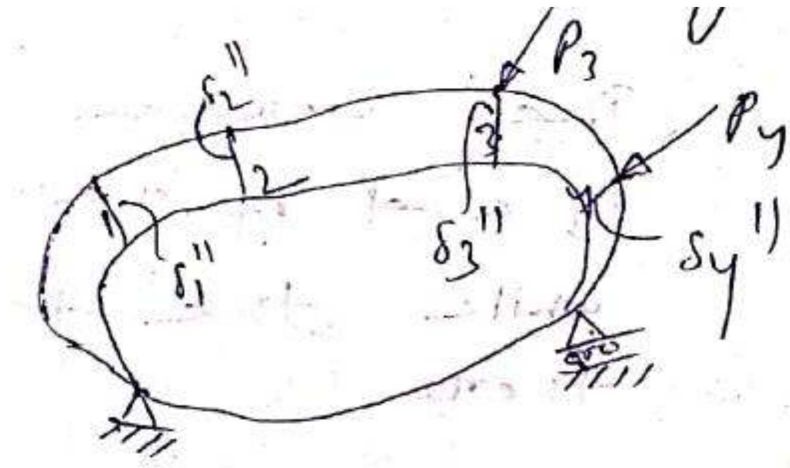
Statement of Betti's Law:

This theorem states that in an elastic structure with unyielding supports and at constant temperature, the work done on a given structure by a system of 1st loading on the corresponding displacements of the 2nd loading is equal to the work done by 2nd loading on the displacements of the 1st loading.



1st loading



2nd loading

Let P_1 and P_2 be the first loading and P_3 and P_4 be the second loading acting at points 1, 2, 3 and 4 respectively.

Let δ_1', δ_2', δ_3', δ_4' be the deflection at points 1, 2, 3 & 4 in the directions of the respective loads due to first loading.

and δ_1'', δ_2'', δ_3'' and δ_4'' be the deflection due to second loading in direction of the respective loads.

If the forces P_1, P_2, P_3 and P_4 be applied gradually and simultaneously at points 1, 2, 3 and 4 respectively, the deflection at the four points will be $(\delta_1' + \delta_1'')$, $(\delta_2' + \delta_2'')$, $(\delta_3' + \delta_3'')$ and $(\delta_4' + \delta_4'')$.

The work done when only P_1 and P_2 are applied $= U_1$

$$U_1 = \frac{1}{2} P_1 \delta_1' + \frac{1}{2} P_2 \delta_2' \quad \text{--- (1)}$$

The workdone on the structure when P_3 and P_4 are gradually applied, while the forces P_1 and P_2 are still there.

$$U_2 = \frac{1}{2} P_3 \delta_3'' + \frac{1}{2} P_4 \delta_4'' + P_1 \delta_1'' + P_2 \delta_2'' \quad \text{— (2)}$$

The workdone when P_1, P_2, P_3 & P_4 are gradually applied

$$U = \frac{1}{2} P_1 (\delta_1' + \delta_1'') + \frac{1}{2} P_2 (\delta_2' + \delta_2'') + \frac{1}{2} P_3 (\delta_3' + \delta_3'') + \frac{1}{2} P_4 (\delta_4' + \delta_4'').$$

$$\boxed{U = U_1 + U_2}$$

$$\frac{1}{2}P_1(\delta_1'+\delta_1'')+\frac{1}{2}P_2(\delta_2'+\delta_2'')+\frac{1}{2}P_3(\delta_3'+\delta_3'')$$
$$+\frac{1}{2}P_4(\delta_4'+\delta_4'')=\frac{1}{2}P_1\delta_1'+\frac{1}{2}P_2\delta_2'+\frac{1}{2}P_3\delta_3''$$
$$+\frac{1}{2}P_4\delta_4''+P_1\delta_1''+P_2\delta_2''$$

After simplification it is obtained as:

$$\frac{1}{2}P_1\delta_1''+\frac{1}{2}P_2\delta_2''=\frac{1}{2}P_3\delta_3'+\frac{1}{2}P_4\delta_4'$$

$$\boxed{P_1\delta_1''+P_2\delta_2''=P_3\delta_3'+P_4\delta_4'}$$

Thus work done by 1st loading on the deflection caused by 2nd loading is equal to the work done by 2nd loading on the deflection caused by the 1st loading.

Thanks

Castigliano's Theorem

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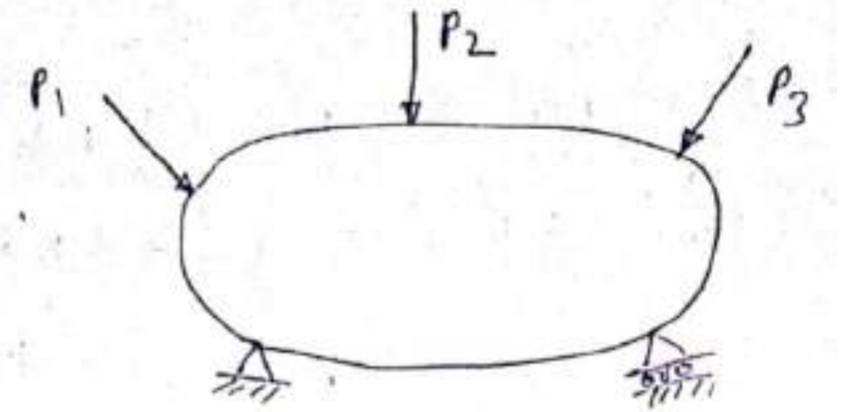
Castigliano's First theorem

- The first theorem of Castigliano states that the partial derivative of the total strain energy in any structure with respect to applied force or moment gives the displacement or rotation respectively at the point of application of the force or moment in the direction of the applied force or moment.

$$\frac{dU}{dP_1} = \Delta_1 \qquad \frac{dU}{dM_2} = \theta_2$$

Proof of the First Theorem



Consider a body subjected to forces P_1, P_2 and P_3 as shown in figure.

let the displacements be Δ_1, Δ_2 and Δ_3 in the directions of P_1, P_2 and P_3 respectively.

Strain energy stored will be equal to

$$U = \frac{P_1 \Delta_1}{2} + \frac{P_2 \Delta_2}{2} + \frac{P_3 \Delta_3}{2}$$

$$\therefore \boxed{2U = P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3} \quad \text{— (1)}$$

Now let the force P_1 be increased by an amount δP_1. This increment of force will cause additional displacements in directions of P_1, P_2 and P_3.

These displacements in direction of Δ_1, Δ_2 and Δ_3 be

$$\left.\begin{aligned} \delta \Delta_1 &= \frac{\partial \Delta_1}{\partial P_1} \cdot \delta P_1 \\ \delta \Delta_2 &= \frac{\partial \Delta_2}{\partial P_1} \cdot \delta P_1 \\ \delta \Delta_3 &= \frac{\partial \Delta_3}{\partial P_1} \cdot \delta P_1 \end{aligned}\right\}$$

The extra energy stored will be

$$\frac{\partial U}{\partial P_1} \cdot \delta P_1 = P_1 \cdot \delta \Delta_1 + P_2 \cdot \delta \Delta_2 + P_3 \cdot \delta \Delta_3 \quad \text{— (2)}$$

Put the value of $\delta\Delta_1$, $\delta\Delta_2$ and $\delta\Delta_3$ in equation (1) we get

$$\frac{\partial U}{\partial P_1}\cdot \delta P_1 = P_1 \cdot \frac{\partial \Delta_1}{\partial P_1}\cdot \delta P_1 + P_2 \cdot \frac{\partial \Delta_2}{\partial P_1}\,\delta P_1 + P_3 \cdot \frac{\partial \Delta_3}{\partial P_1}\cdot \delta P_1$$

$$\boxed{\therefore \frac{\partial U}{\partial P_1} = P_1 \cdot \frac{\partial \Delta_1}{\partial P_1} + P_2 \cdot \frac{\partial \Delta_2}{\partial P_1} + P_3 \cdot \frac{\partial \Delta_3}{\partial P_1}} \quad \text{--- (3)}$$

Differentiating equation (1) w.r.T. P_1

$$2 \cdot \frac{\partial U}{\partial P_1} = \Delta_1 + P_1 \cdot \frac{\partial \Delta_1}{\partial P_1} + P_2 \cdot \frac{\partial \Delta_2}{\partial P_1} + P_3 \cdot \frac{\partial \Delta_3}{\partial P_1} \quad \text{--- (4)}$$

Subtracting (3) from (4) we get,

$$\boxed{\frac{\partial U}{\partial P_1} = \Delta_1}$$ Proof of 1st theorem.

Thus the Partial derivative of Strain energy with respect to P_1 gives displacement in the direction of P_1.

Castigliano's Second Theorem

- The second theorem of castigliano states that the work done by external forces in a structure will be minimum.

- The Theorem is very much useful in analysis of statically indeterminate structures.

Let W = Work done by external forces on a structure

U = Strain energy stored in the structure

W_1 = Work done by reactive forces

Strain Energy = $U = W + W_1$

$$W = U - W_1$$

By Castigliano's 2nd theorem 'W' should be minimum.

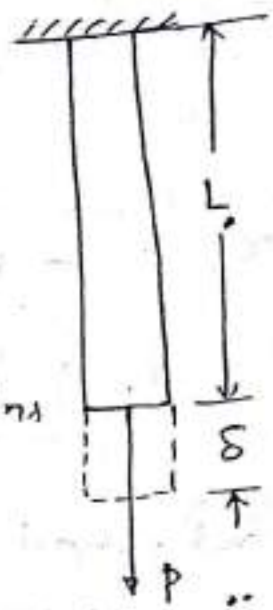
Thus the partial derivative of the work done with respect to external forces will be zero.

- In case the supports are unyielding, the work done by reactive forces will be zero.
- Strain energy stored is equal to the work done by external forces will be minimum.
- Thus the partial derivative of strain energy with respect to redundant reaction will be zero.
- Castigliano's First theorem helps in determining deflection of a structure and the Second theorem helps in determining redundant reaction components.

Law of Conservation of Energy

This is the basic law of physics. energy is neither created nor can be destroyed.

If a structure and external loads acting on it are isolated, such that they neither receive nor give out energy, then the total energy of the system remains constant.



①

A typical application of the law of conservation of energy can be applied to a bar subjected to an axial pull 'P' gradually applied as shown in fig ①.

when equilibrium is reached, it will be found that the bar has extended by an amount 'δ'.

Considering that the process is adiabatic (heat is neither supplied nor taken out)

According to the law of conservation energy

loss of potential energy $= \frac{1}{2} P\delta$.

The strain energy stored in the bar is given by

$$\boxed{W_i = \int_0^L \frac{P^2\,dx}{2AE}} \quad \text{--- (1)}$$

where W_i = strain energy stored in the body or internal work

let

$$\boxed{W_e = \frac{1}{2} P\delta} \quad \text{--- (2)}$$

where We = external work done.

$$\therefore \; W_e + W_i = -\frac{1}{2} P\delta + \int_0^L \frac{P^2 dx}{2AE} = 0. \quad \text{—(3)}$$

< The minus sign for external work is to take into account the loss of potential energy >.

or $$-W_e + W_i = 0. \quad \text{—(4)}$$

or, $$\boxed{W_e = W_i} \quad \text{—(5)}$$

$$\boxed{\text{External work done} = \text{Internal strain energy}}$$

Thanks

Deflection by Castigliano's Method

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Deflection by Castigliano's Method:

Castigliano's theorem may be represented by

$$\frac{dU}{dP_i} = \Delta_i, \quad \frac{dU}{dM_j} = \theta_j \qquad U = \int_0^L \frac{M^2}{2EI}dx$$

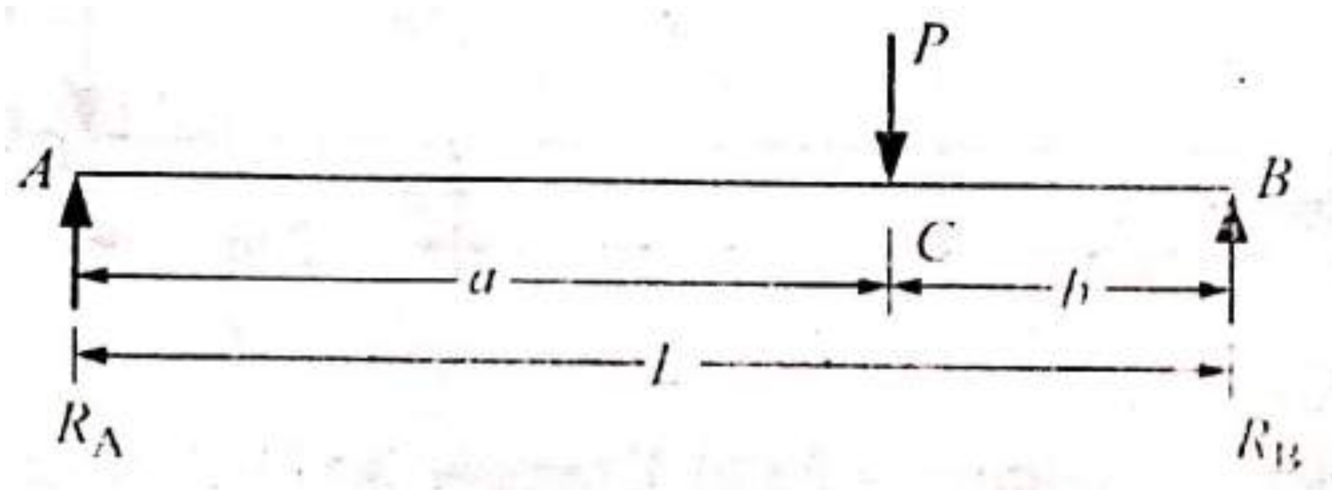
where U = total strain energy

P_i, M_j – loads

Δ_i, θ_j – deflections.

- If a load is acting at a point and is in the desired direction, the general expression for bending moment to cover the entire structure is to be find out.
- The strain energy for the entire structure is differentiated with respect to load (*P* = Load or *M* = Moment) to get the desired deflection.
- If the load is not acting, a dummy load (*P* or *M*) is applied and then the bending moment expressions is to be find out.
- If dummy load is used, First differentiate w.r.t the dummy load, then substitute dummy load as zero and then integrate w.r.t 'x' .

Q1. A simply supported beam of span '*L*', carries a concentrated load '*P*' at a distance '*a*' from the left hand side as shown in Figure. Using Castigliano's theorem determine the deflection under the load. Assume uniform flexural rigidity.



First determine the reaction by taking moment from any one support,

- Reaction at A, $R_A = \frac{Pb}{L}$
- Reaction at B, $R_B = \frac{Pa}{L}$
- Find out the expression for moment in a Tabular form for portion BC and then AC.

Portion	AC	CB
Origin	A	B
Limit	$0-a$	$0-b$
M	$\frac{Pb}{L}x$	$\frac{Pa}{L}x$
Flexural Rigidity	EI	EI

The strain energy of the Beam = $U = \int_0^L \frac{M^2}{2EI} dx$

$$U = \int_0^a \left(\frac{Pb}{L} x \right)^2 \times \frac{1}{2EI} dx + \int_0^b \left(\frac{Pb}{L} x \right)^2 \times \frac{1}{2EI} dx$$

$$= \left[\frac{P^2 b^2}{L^2} \times \frac{1}{6EI} x^3\right]_0^a + \left[\frac{P^2 a^2}{L^2} \times \frac{1}{6EI} x^3\right]_0^b$$

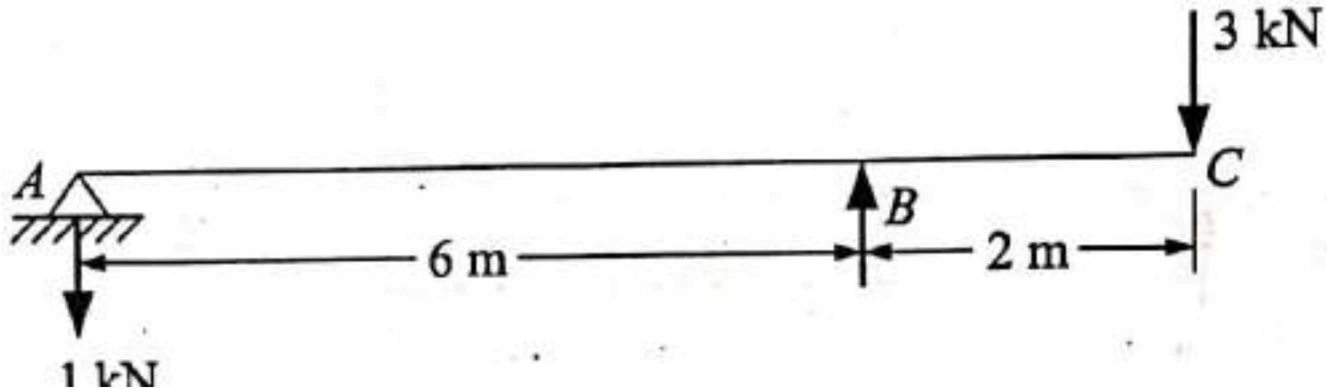
$$= \frac{P^2 b^2 a^3}{6EIL^2} + \frac{P^2 a^2 b^3}{6EIL^2}$$

$$= \frac{P^2 a^2 b^2}{6EIL^2}(a+b)$$

$$= \frac{P^2 a^2 b^2}{6EIL}, \text{ Since, } a + b = L$$

$$\Delta_C = \frac{\delta U}{\delta P} = \frac{Pa^2 b^2}{3EIL}$$

Q2. Determine the vertical deflection at the free end and rotation at '*A*' in the over hanging beam shown in Figure. Use Castigliano's theorem. Assume uniform flexural rigidity.



Deflection at 'C' = Δ_c

- Taking force P = 3 kN and moment about *A*,

$$R_B \times 6 = P \times 8$$

$$R_B = \frac{4}{3}P \uparrow$$

$$R_A = \frac{P}{3} \downarrow$$

Bending moment expression for over hanging beam for portion AB and BC is noted in the Tabular form.

Portion	*AB*	*BC*
Origin	*A*	*C*
Limit	0–6	0–2
M	$\frac{-P}{3}x$	$-Px$
Flexural Rigidity	*EI*	*EI*

$$U = \int \frac{M^2}{2EI} dx$$

$$= \int_0^6 \frac{P^2 x^2}{9} \times \frac{1}{2EI} dx + \int_0^2 \frac{P^2 x^2}{2EI} dx$$

$$= \frac{P^2}{18EI}\left[\frac{x^3}{3}\right]_0^6 + \left[\frac{P^2 x^3}{6EI}\right]_0^2$$

$$= \frac{4P^2}{EI} + \frac{4}{3} \times \frac{P^2}{EI}$$

$$= \frac{5.333P^2}{EI}$$

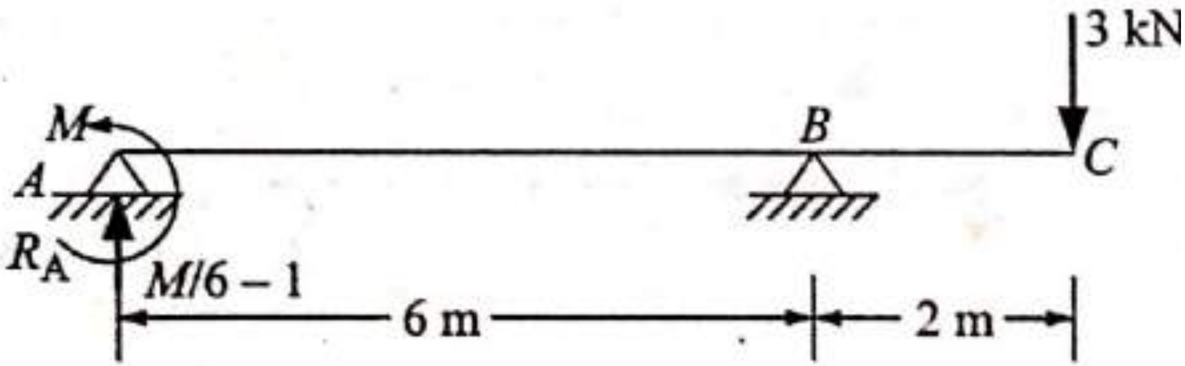
$$\Delta_C = \frac{dU}{dP} = \frac{10.667P}{EI}$$

Substituting $P = 3$ kN, we get

$$\Delta_C = \frac{32}{EI}$$

Rotation at A = θ_A

- Apply dummy moment, '*M*' at *A* as shown in Figure



$$\Sigma M_B = 0, \text{ gives}$$

$$R_A = \frac{M-6}{6} = \frac{M}{6} - 1$$

Portion	AB	BC
Origin	A	C
Limit	0 – 6	0 – 2
M	$\left(\frac{M}{6}-1\right)x - M$	$-3x$

$$U=\int_0^6\left[\left(\frac{M}{6}-1\right)x-M\right]_0^2\frac{1}{2EI}dx+\int_0^2\frac{(-3x)^2}{2EI}dx$$

$$\frac{dU}{dM}=\int_0^6 2\left[\left(\frac{M}{6}-1\right)x-M\right]\left(\frac{x}{6}-1\right)\frac{dx}{2EI}+0$$

Since, '*M*' is a dummy moment, its value is substituted as zero, and then integrated

$$\frac{dU}{dM}=\theta_A=\frac{1}{EI}\int_0^6(-x)\left(\frac{x}{6}-1\right)dx$$

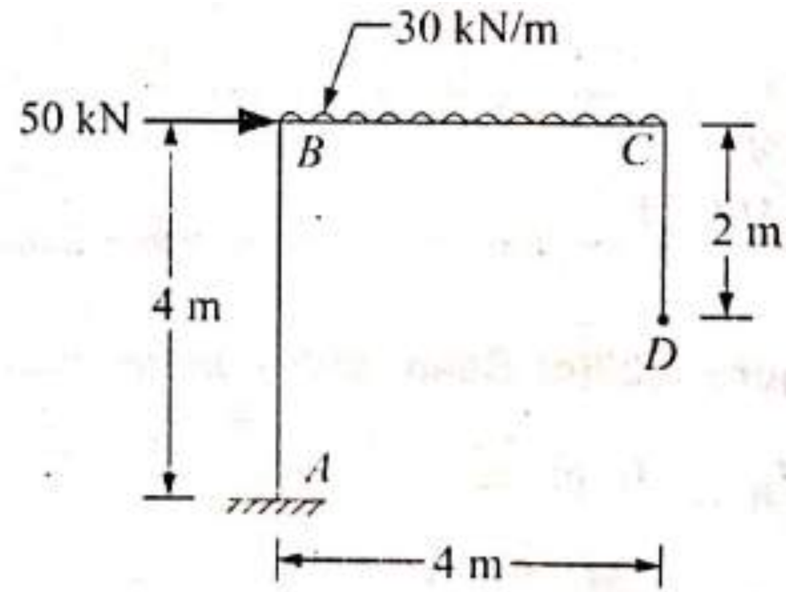
$$=\frac{1}{EI}\int_0^6\left(-\frac{x^2}{6}+x\right)dx$$

$$=\frac{1}{EI}\left(-\frac{x^3}{18}+\frac{x^2}{2}\right)_0^6 dx$$

$$=\frac{6}{EI}$$

Note: First differentiate w.r.t the dummy load, then substitute dummy load as zero and then integrate w.r.t 'x'.

Q2. Determine the vertical and horizontal deflection at the free end '*D*' in the frame shown in Figure. Use Castigliano's theorem. Take $EI = 12 \times 10^{13}$ Nmm^2.



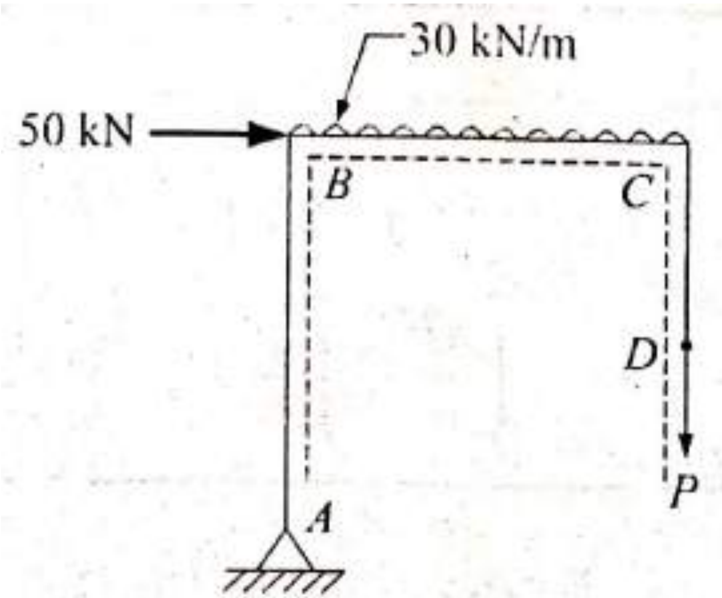


Figure 1: Frame with dummy vertical load '*P*' at '*D*'

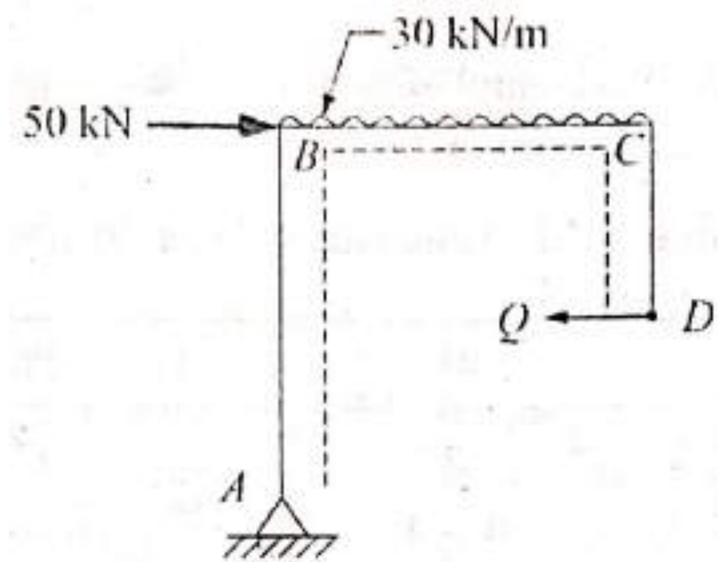


Figure 2: Frame with dummy horizontal load '*Q*' at '*D*'

Vertical Deflection:

- Since, there is no load at '*D*' in vertical direction, a dummy load '*P*' is applied at '*D*' in vertical direction in addition to given loads as shown in Figure 1. The moment expressions are presented in a tabular form.

Portion	*AB*	*BC*	CD
Origin	*B*	*C*	*D*
Limit	0 – 4	0 – 4	0 – 2
M	$-(4P+240+50x)$	$-(Px+15x^2)$	0
Flexural Rigidity	*EI*	*EI*	*EI*

Strain energy $U = \int \frac{M^2}{2EI}dx$

$$= \int_0^4 \frac{(4P+240+50x)^2}{2EI}dx + \int_0^4 \frac{\left(Px+15x^2\right)^2}{2EI}dx + 0$$

$$\Delta = \frac{\delta U}{\delta P} = \int_0^4 2\frac{(4P+240+50x)}{2EI}(4)\,dx + \int_0^4 2\frac{\left(Px+15x^2\right)x}{2EI}dx$$

Since, P is dummy load, substitute $P = 0$

$$\Delta_D = \int_0^4 \frac{4(240+50x)}{EI}dx + \int_0^4 \frac{15x^3}{EI}dx$$

$$= \frac{4}{EI}\left[240x+25x^2\right]_0^4 + \frac{15}{EI}\left(\frac{x^4}{4}\right)_0^4 = \frac{6400}{EI}$$

Now,

$$EI = 12 \times 10^{13} \text{ Nmm}^2$$
$$= 12 \times 10^4 \text{ kNm}^2$$

$$\therefore \quad \Delta_{DV} = \frac{6400}{12 \times 10^4} = 0.533 \text{ m}$$
$$= 53.33 \text{ mm}$$

Horizontal Deflection:

- Since, there is no load at '*D*' in horizontal direction, a dummy load '*Q*' is applied at '*D*' in horizontal direction in addition to given loads as shown in Figure 2. The moment expressions are presented in a tabular form.

Portion	AB	BC	CD
Origin	B	C	D
Limit	0 – 4	0 – 4	0 – 2
M	$-[Q(2-x)+240+50x)]$	$-(2Q+15x^2)$	Qx
Flexural Rigidity	EI	EI	EI

$$U=\int_0^4 \frac{\left[Q(2-x)+240+50x\right]^2}{2EI}dx+\int_0^4 \frac{\left[\left(2Q+15x^2\right)\right]^2}{2EI}dx+\int_0^2 \frac{Q^2x^2}{2EI}dx$$

$$\Delta_{DH}=\frac{dU}{dQ}=\int_0^4 \frac{2\left[Q(2-x)+240+50x\right](2-x)}{2EI}dx+\int_0^4 \frac{2\left[2Q+15x^2\right]2}{2EI}dx+\int_0^2 \frac{2Qx^2}{2EI}dx$$

Substituting $Q = 0$

$$\Delta_{DH} = \int_0^4 \frac{(240+50x)(2-x)}{EI}dx + \int_0^4 \frac{30x^2}{EI}dx + 0$$

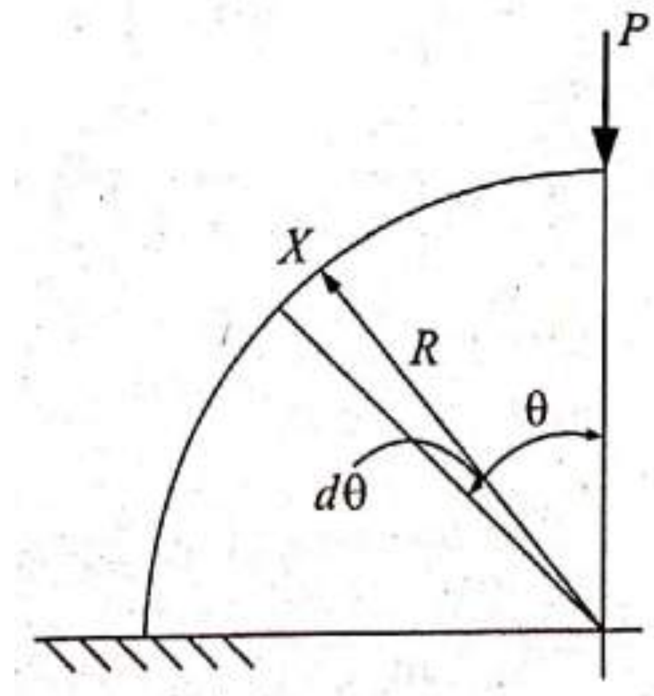
$$= \int_0^4 \frac{\left(480-140x-50x^2\right)}{EI}dx + \int_0^4 \frac{30x^2}{EI}dx$$

$$= \frac{1}{EI}\left[480x - 70x^2 - \frac{50x^3}{3}\right]_0^4 + \frac{1}{EI}\left[10x^3\right]_0^4$$

$$= \frac{373.33}{EI} = \frac{373.33}{12\times10^4} = 0.0031 \text{ m}$$

$$= 3.1 \text{ mm}$$

Q2. A cantilever beam is in the form of a quarter of a circle in the vertical plane and is subjected to a vertical load ‘P’ at its free end as shown in Figure. Find the vertical and horizontal deflections at the free end. Use Castigliano’s theorem. Assume uniform flexural rigidity.



Vertical Deflection of free end:

- Consider the section at ‘x’ as shown in Figure 1. The Bending moment at the section ‘x’ is

$$M = PR \sin \theta$$

Strain energy in the elemental length '$R\,d\theta$' is

$$= \left(\frac{M^2}{2EI}\right) R\,d\theta$$

$$= \frac{P^2 R^2 \sin^2\theta}{2EI} R\,d\theta$$

$$= \frac{P^2 R^3}{2EI} \times \frac{1-\cos 2\theta}{2} d\theta$$

$$U = \int_0^{\pi/2} \frac{P^2 R^3}{2EI} \times \frac{1-\cos 2\theta}{2} d\theta$$

$$= \frac{P^2 R^3}{4EI}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi P^2 R^3}{8EI}$$

$$\Delta_V = \frac{\delta U}{dP} = \frac{\pi P R^3}{4EI}$$

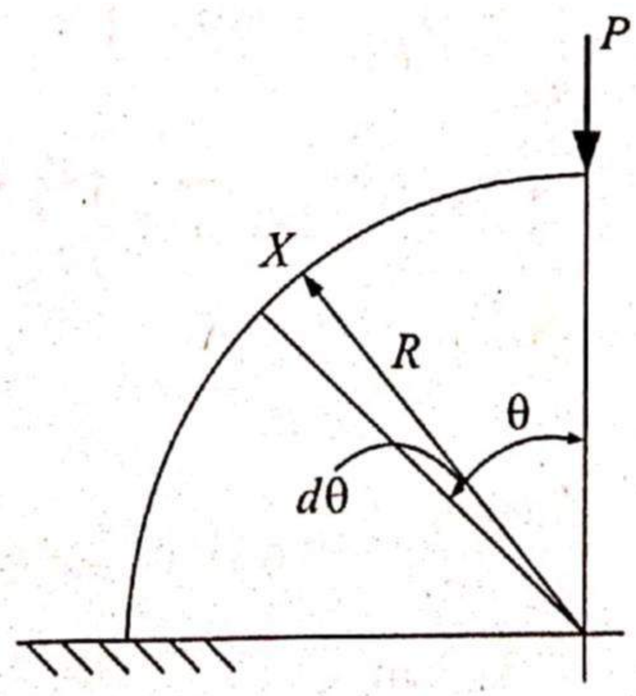


Figure 1: Cantilever curved beam

Horizontal Deflection:

Since, there is no horizontal force at the free end, apply a dummy horizontal force '*Q*', as shown in Figure 2.

The bending moment at section '*x*' is

$$M = PR \sin\theta + QR(1 - \cos\theta)$$

Strain Energy (U) =

$$U = \int_0^{\pi/2} \frac{\left[PR\sin\theta + QR(1-\cos\theta)\right]^2}{2EI} R\, d\theta$$

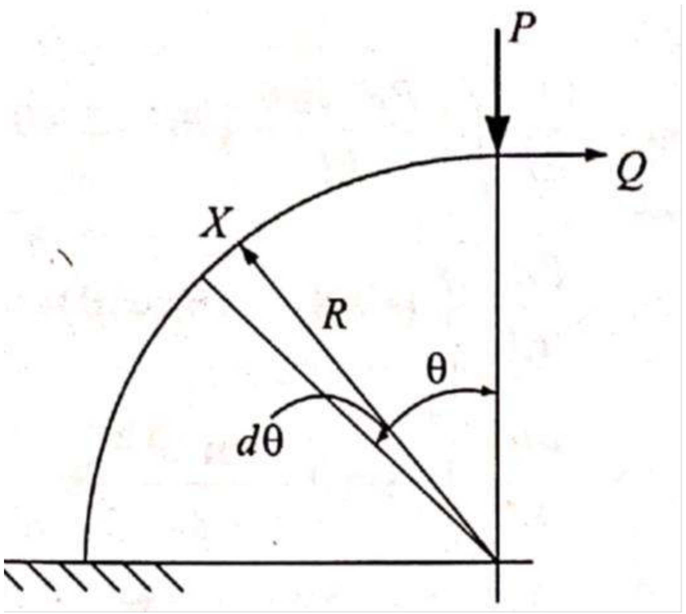


Figure 2: Cantilever curved beam with dummy load '*Q*' at the free end

Horizontal Displacement = Δ_H

$$\Delta_H = \frac{\delta U}{\delta Q} = \int_0^{\pi/2} \frac{\left[PR\sin\theta + QR(1-\cos\theta)\right]}{EI}\left[R(1-\cos\theta)\right] R\, d\theta$$

Substituting, Q = 0, in above equation

$$\Delta_H = \frac{1}{EI}\int_0^{\pi/2}\left(\frac{PR\sin\theta}{EI}\right)[R(1-\cos\theta)]\,R\,d\theta$$

$$= \frac{PR^3}{EI}\int_0^{\pi/2}(\sin\theta-\sin\theta\cos\theta)\,d\theta$$

$$= \frac{PR^3}{EI}\int_0^{\pi/2}\left(\sin\theta-\frac{\sin 2\theta}{2}\right)d\theta$$

$$= \frac{PR^3}{EI}\left(\cos\theta-\frac{\cos 2\theta}{4}\right)_0^{\pi/2}$$

$$= \frac{PR^3}{EI}\left(0+\frac{1}{4}-1+\frac{1}{4}\right)$$

$$= -\frac{PR^3}{2EI}$$

i.e., $\Delta_H = \frac{PR^3}{2EI}$, towards support

Thanks

Virtual Work Method

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Introduction

- Virtual work methods are the most direct methods for calculating deflections in statically determinate and indeterminate structures. This principle can be applied to both linear and nonlinear structures. The principle of virtual work as applied to deformable structure is an extension of the virtual work for rigid bodies.

- This may be stated as: if a rigid body is in equilibrium under the action of a *F*- system of forces and if it continues to remain in equilibrium if the body is given a small (virtual) displacement, then the virtual work done by the *F*-system of forces as 'it rides' along these virtual displacements is zero.

Principle of Virtual Work

- When a real force 'F' under goes a real displacement 'Δ', then the product FΔ is called Real Work.
- Suppose, the force or the displacement is virtual or imaginary, we denote the virtual force as 'Q' and the virtual displacement as 'δ'.
- The product QΔ or Fδ is called Virtual Work in which F, Δ are real and Q, δ are virtual.
- The virtual forces or displacements may be finite or infinitesimal.
- As per the principle of virtual work, the total virtual work done by a set of forces is undergoing a set of displacements is zero provided the following conditions are satisfied.
- Any one of the parameter, i.e. a set of forces or a set of displacements is virtual.
- Whether the set of forces is real or virtual, it must satisfy the equilibrium conditions. Similarly, the set of displacements must be compatible with the prescribed boundary conditions.

There are two versions of principle of virtual work.

- Principle of virtual forces
- Principle of virtual displacements

Principle of virtual forces:

The principle of virtual forces states that the total work done by a set of virtual forces both external and internal in under going a set of real displacements is zero, provided that the set of virtual forces is in equilibrium and the set of real displacements is compatible with the given boundary conditions.

The principle of virtual forces may be written as:

$$W_Q = \sum \int Q \, d\Delta - \sum Q_i \, d\delta = 0$$

$$W_Q = \sum Q \, \Delta - \sum Q_i \, \delta = 0$$

$$\mathbf{\sum Q \, \Delta = \sum Q_i \, \delta}$$

Where, W_Q = Total virtual work done

Q = Applied external virtual force

Q_i = Induced internal virtual force

Δ = External deformation

δ = Internal deformation

Principle of virtual displacements:

- The principle states that the virtual work done by a set of real forces in under going a set of virtual displacements is zero provided that a set of real forces is in equilibrium and the virtual displacement is compatible with the given restraints.
- The Principle of virtual displacement may be written as:

$$W_Q = \sum F_e \delta - \sum F_i \delta_i = 0$$

or $$\boldsymbol{\sum F_e \delta = \sum F_i \delta_i}$$

Where, F_e = Real external forces

F_i = Real internal stresses

δ = Virtual external displacements

δi = Virtual internal deformation

Thanks

Unit Load Method

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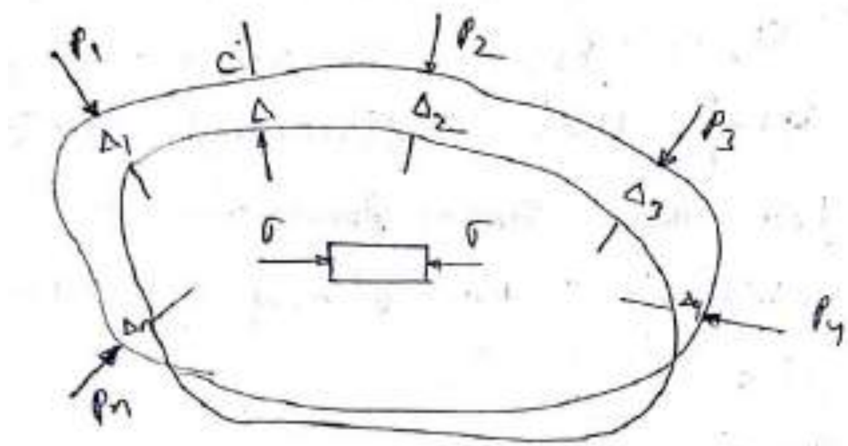
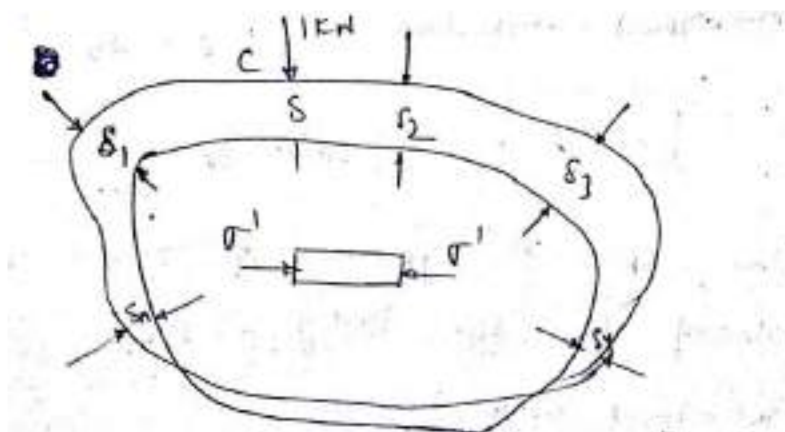
Proof of Unit Load Method

Consider the body shown in fig. which is subjected to forces $P_1, P_2, P_3, P_4 \ldots \ldots P_n$ applied gradually.

Let the displacement under load points be $\Delta_1, \Delta_2, \Delta_3 \ldots \Delta_n$ and at point 'C' be Δ.



Fig①



Fig②

and Strain energy stored $= \int \frac{1}{2}\sigma \cdot e \cdot dv$.

where σ = Stress

e = Strain in the element.

$\therefore$ External work done = Strain energy stored

$$\frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta_2 P_2 + \frac{1}{2}\Delta_3 P_3 + \cdots + \frac{1}{2}\Delta_n P_n = \int \frac{1}{2}\sigma \cdot e\, dv \quad \text{--- (1)}$$

Now, Consider the same body subjected to an unit load applied gradually at 'c' when it is free of system of 'p' forces.

Let the displacements at 1, 2, 3 ... n be $\delta_1, \delta_2, \delta_3, \ldots \delta_n$ respectively and the displacement at 'c' be 'δ'.

Let the stress produced in the element be σ' and the strain be e'.

External work done $= \frac{1}{2} \times 1 \times \delta$.

Internal work done $= \int \frac{1}{2} \sigma' e' \, dv$.

$$\frac{1}{2} \times 1 \times \delta = \int \frac{1}{2} \sigma' e' . dv \quad \text{—— (2)}$$

Now, if 'P' system of forces is applied to the body as shown in fig (2)

External work done

$$= \frac{1}{2} \Delta_1 P_1 + \frac{1}{2} \Delta_2 P_2 + \cdots + \frac{1}{2} \Delta_n P_n + 1 \times \Delta$$

Since, unit load is already acting.

Internal work done $= \int \frac{1}{2}\sigma \cdot e \, dv + \int \sigma' e \, dv$,

Since the stress σ' is acting throughout the deformation.

External work done = Internal work done

$$\frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta_2 P_2 + \cdots + \frac{1}{2}\Delta_n P_n + 1 \times \Delta$$

$$= \int \frac{1}{2}\sigma \cdot e \, dv + \int \sigma' e \, dv \quad \text{--- (7)}$$

Subtracting equation (1) from (3)

$$1 \times \Delta = \int \sigma' e \, dv$$

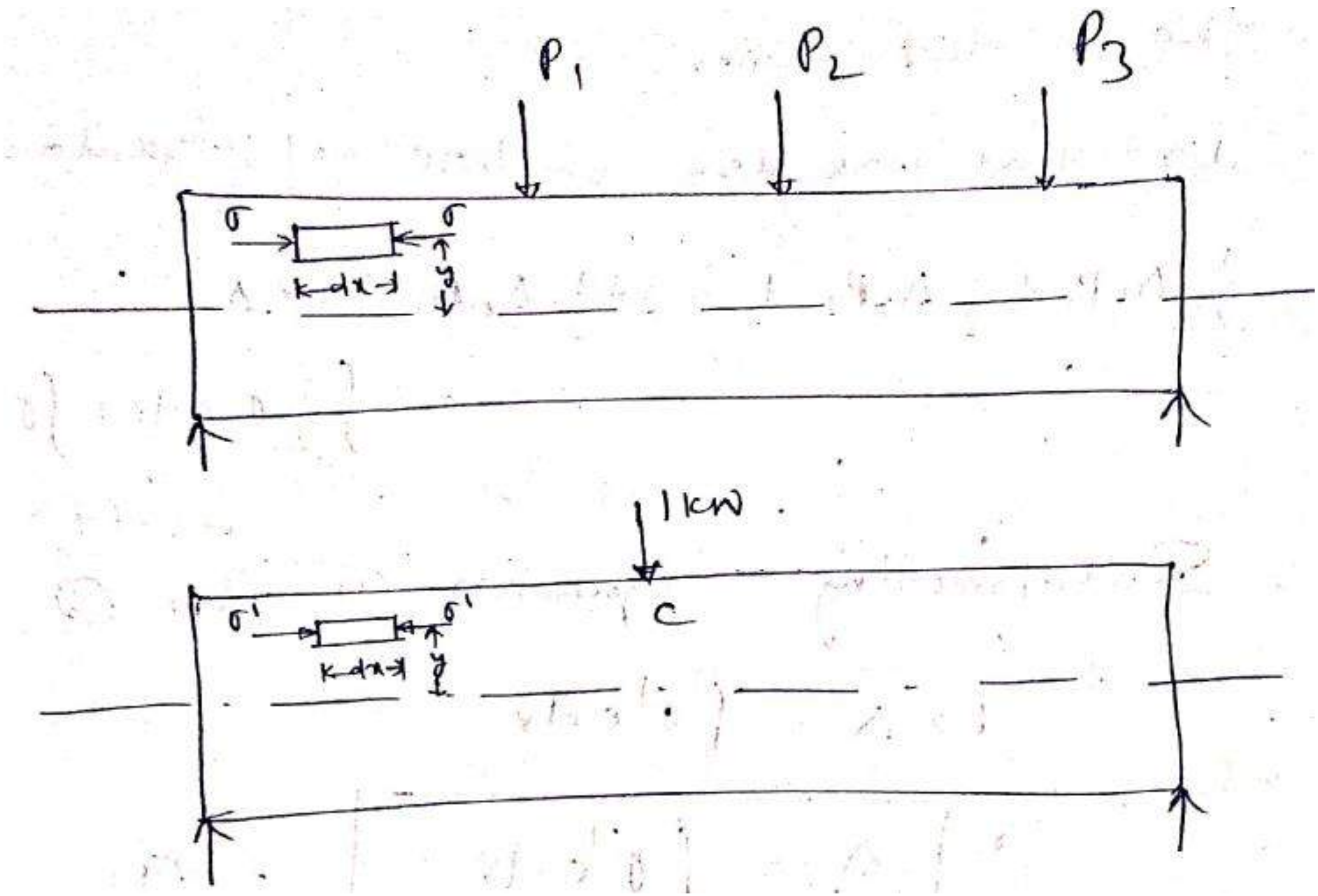
or $$\boxed{\Delta = \int \sigma' e \, dv} \quad \text{--- (4)}$$

where Δ = deflection at point where unit load is applied and is measured in the direction of unit load,

σ' = Stress in an element due to unit load.

and e = Strain in the element due to given load system.

Application of Unit Load Method to Beam Deflection



Consider the beam subjected to a system of 'p' forces.

The stress in the element at a distance 'y' from neutral axis is

$$\boxed{\sigma = \frac{M}{I} y}$$

where 'M' = Moment acting at the section

Strain in the element due to given system of forces is

$$\boxed{e = \frac{M}{EI} y}$$

Let 'm' is the moment at the section due to unit load acting at 'c'

Then stress = $\boxed{\sigma' = \frac{M}{I} \cdot y}$

From equation (1)

Deflection = $\boxed{\Delta = \int \sigma' e \, dV}$

Putting the value of σ' & e in the above equation

$$\Delta = \int \frac{M}{I} \cdot y \cdot \frac{M}{EI} y \cdot dV$$

$$\Delta = \int \frac{mM}{EI^2} \cdot y^2 \, dv$$

$$= \int_0^L \frac{Mm}{EI^2} \left[\int_0^A y^2 dA \right] dx$$

But $\boxed{\int_0^A y^2 dA = I}$

$$\therefore \Delta = \int_0^L \frac{Mm}{EI^2} \cdot I \cdot dx$$

$$\boxed{\Delta = \int_0^L \frac{Mm}{EI} dx} \quad \text{--- (5)}$$

The equation (5) used to find out the deflection at any point 'c'. It needs bending moment due to a given load system and unit load acting at 'c'.

This procedure is applicable to rigid frames also, where only flexure effect is considered (i.e in the analysis in which the effect of axial & shear forces are neglected)

Thanks

Deflection by Unit Load Method

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Deflection by Unit Load Method

- This method is applicable to beam and rigid frame where only flexural effect is considered.
- In the analysis, the effect of axial force and shear forces are neglected.
- The deflection at any point can be find out by:

$$\Delta = \int_0^L \frac{Mm}{EI}\,dx$$

Where, M = Bending moment at the section due to the external forces
m = Bending moment at the section due to unit loading
E = Modulus of Elasticity
I = Moment of Inertia of the section

Q1. Determine the deflection at the free end of the over hanging beam shown in Figure by unit load method.

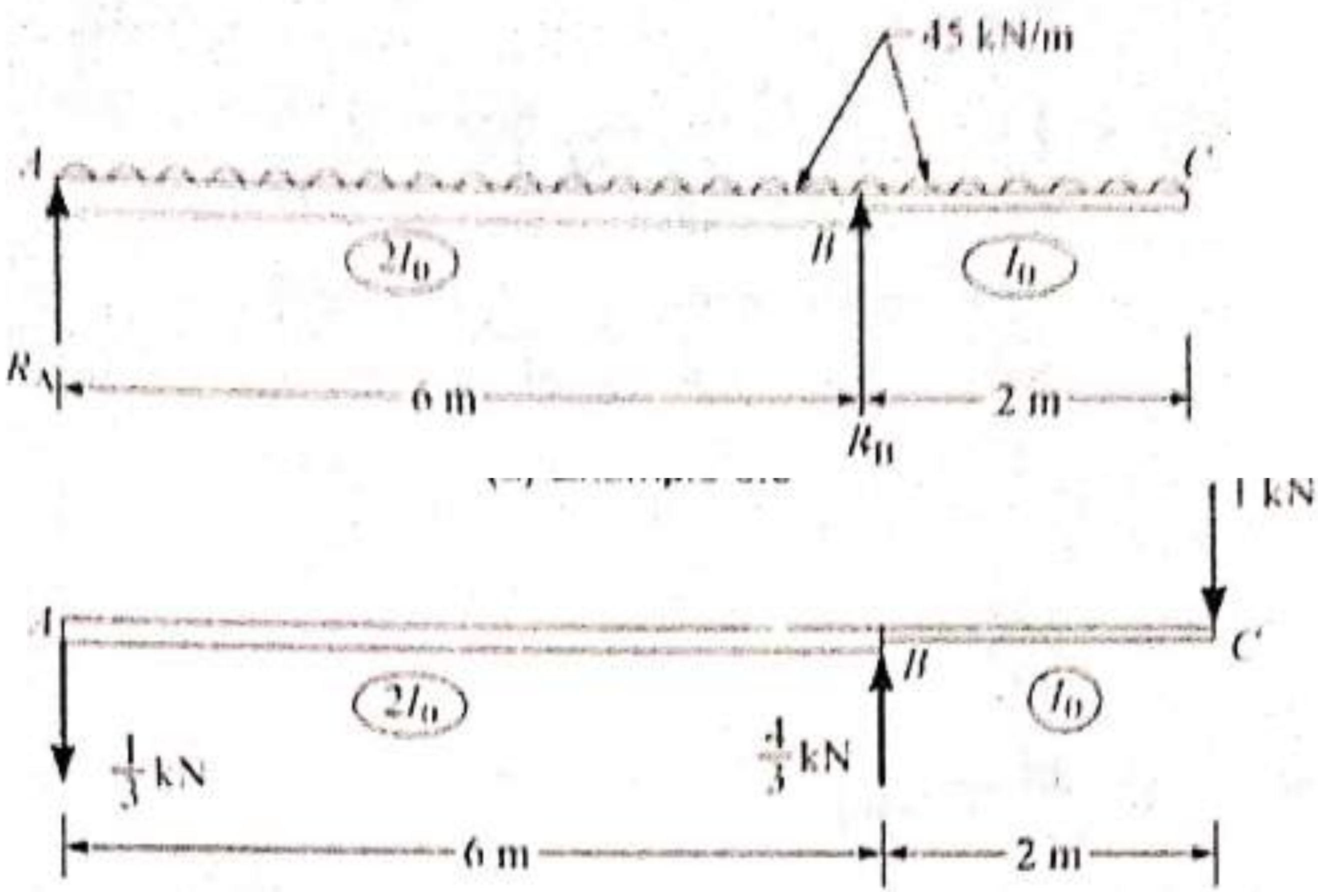


Figure 1: Beam with unit load at '*C*'

- Find out the reactions due to external forces, taking moment about A

$$\Sigma M_A = 0, \text{ gives}$$

$$R_B \times 6 = 45 \times 8 \times 4$$

$$R_B = 240 \text{ kN}$$

$$\Sigma V = 0, \text{ gives}$$

$$R_A = 45 \times 8 - 240 = 120 \text{ kN}$$

- Find out the reactions, when unit load acting at '*C*'

$$R_B = \frac{1 \times 8}{6} = 1.333 \text{ kN}$$

$$R_A = 0.333 \text{ kN} \downarrow$$

- Taking sagging moment as positive and hogging moment as negative, find out the expressions for moments in various portions of the beam due to external loading and unit force where the deflection is to be determined in a Tabular form.

Portion	AB	BC
Origin	A	C
Limit	0 – 6	0 – 2
M	$120x - \frac{1}{2} \times 45x^2$	$-\frac{1}{2} \times 45x^2$
m	$-0.333x$	$-x$
I	$2I_0$	I_0

$$\Delta_c = \int_0^6 \frac{(120x - 22.5x^2)(-0.333x)dx}{E2I_o} + \int_0^2 \frac{(-22.5x^2)(-x)dx}{EI_o}$$

$$= \int_0^6 \frac{(-20x^2 + 3.7x^3)dx}{EI_o} + \frac{1}{EI_o}\int_0^2 22.5x^3 dx$$

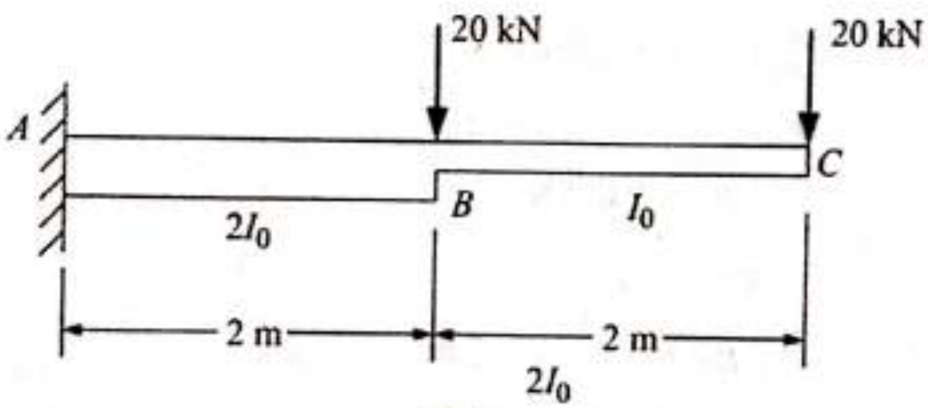
$$= \frac{1}{EI_o}\left[-\frac{20x^3}{3} + \frac{3.75x^4}{4}\right]_0^6 + \frac{1}{EI_o}\left[\frac{22.5x^4}{4}\right]_0^2$$

$$= \frac{1}{EI_o}\left[-\frac{20\times 6^3}{3} + \frac{3.75\times 6^4}{4} + \frac{22.5\times 2^4}{4}\right]$$

$$= -\frac{135}{EI_o}$$

$$= \frac{135}{EI_o}, \text{ upward}$$

Q2. Determine the deflection and rotation at the free end of the cantilever beam shown in Figure by unit load method. Given $E = 200000\ \text{N/mm}^2$ and $I = 12 \times 10^6\ \text{mm}^4$



- Find out the deflection and rotation at the free end of the cantilever beam, apply unit load for deflection and unit moment for rotation at the free end of the beam as shown in Figure.

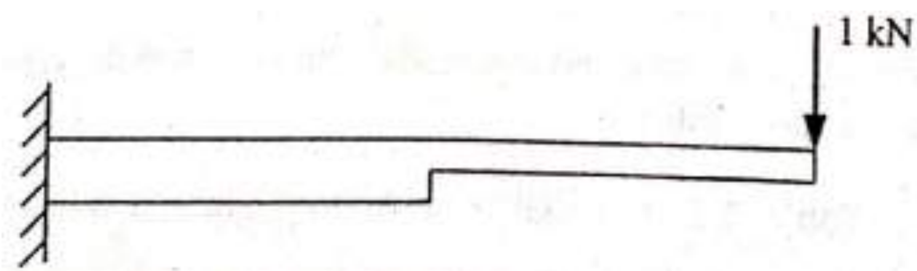


Figure 1: Beam with unit vertical load at '*C*'

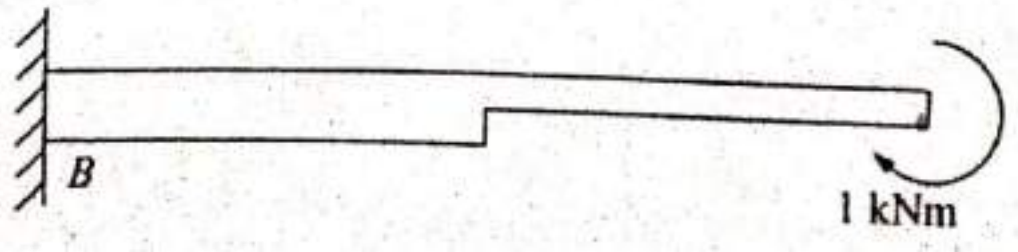


Figure 2: Beam with unit moment at '*C*'

- The bending moment expressions can be calculated by
- M for external given load, m_1 for unit vertical load at 'C' and m_2 for unit moment at 'C' for various portion of cantilever beam and tabulated below.

Portion	*CB*	*BA*
Origin	*C*	*B*
Limit	0 – 2	0 – 2
M	$-20x$	$-[\,20\,(2+x)+20x\,]$
m_1	$-x$	$-(x+2)$
m_2	-1	-1
I	I_0	$2I_0$

Vertical deflection at 'C' = $\Delta = \int_0^L \frac{Mm_1}{EI} dx$

$$= \int_0^2 \frac{(-20x)(-x)}{EI_o} dx + \int_0^2 \frac{[20(2+x)+20x](x+2)}{E2I_o} dx$$

$$= \int_0^2 \frac{20x^2}{EI_o} dx + \int_0^2 \frac{(40x+40)(x+2)}{2EI_o} dx$$

$$= \left[\frac{20}{3}\frac{x^3}{EI_o}\right]_0^2 + \frac{1}{2EI_o}\left[\frac{40x^3}{3} + \frac{120x^2}{2} + 80x\right]_0^2$$

$$= \frac{53.333}{EI_o} + \frac{1}{EI_o}[253.333]$$

$$= \frac{306.67}{EI_o}$$

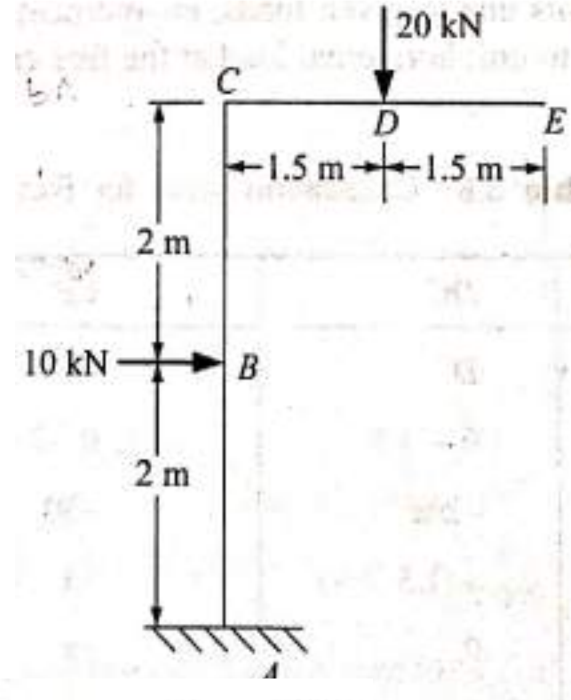
Rotation at 'C' = $\theta_c = \int_0^L \frac{Mm_2}{EI} = \int_0^2 \frac{(20x)}{EI_o}dx + \int_0^2 \frac{(40x+40)1}{E2I_o}dx$

$$= \left[\frac{20}{EI_o}\frac{x^2}{2}\right]_0^2 + \frac{1}{EI_o}\left[\frac{40x^2}{2} + 40x\right]_0^2$$

$$= \frac{40}{EI_o} + \frac{160}{2EI_o}$$

$$= \frac{120}{EI_o}$$

Q3. Determine the vertical and horizontal deflection at the free end of the bent shown in Figure by unit load method. Assume uniform flexural rigidity *EI* throughout.



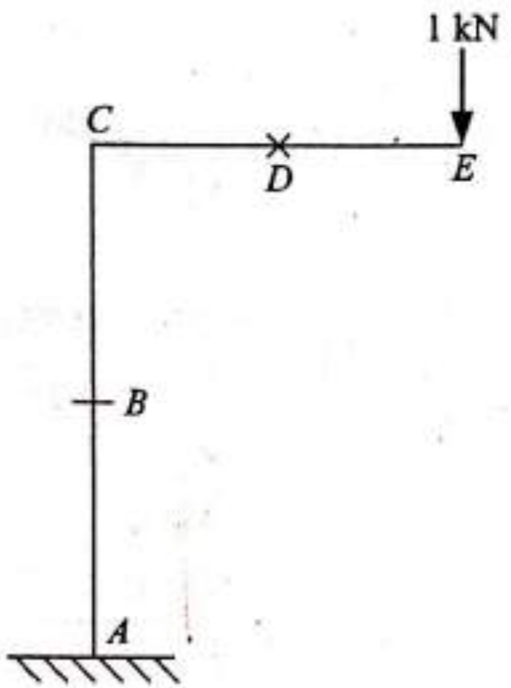


Figure 1: Frame with unit vertical load at '*E*'

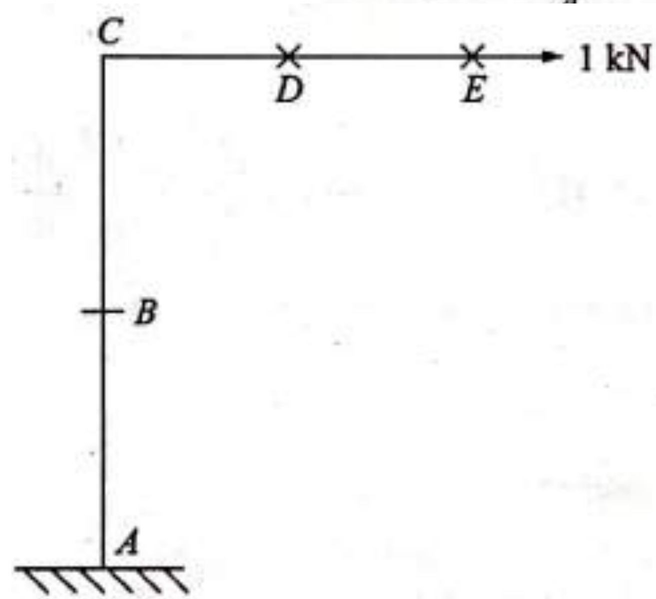


Figure 2: Frame with unit horizontal load at '*E*'

- Find out the expressions in Tabular form for moment 'M' due to external loads, m_1 due to the unit vertical load present at the free end (Figure 1) and m_2 due to the unit horizontal load present at the free end (Figure 2) of the bent.

Portion	ED	DC	CB	BA
Origin	E	D	C	B
Limit	0 – 1.5	0 – 1.5	0 –2	0 – 2
M	0	$-20x$	-30	$-30 - 10x$
m_1	x	$-(1.5 + x)$	-3	-3
m_2	0	0	$-x$	$-(x + 2)$
Flexural Rigidity	EI	EI	EI	EI

Note: Moment carrying tension on dotted side is taken as positive

Vertical deflection at 'E' = Δ_{EV}

$$EI\Delta_{EV} = \int Mm_1\, dx$$

$$= 0+\int_0^{1.5} 20x(1.5+x)\,dx+\int_0^{2} 90\,dx+\int_0^{2} (90+30x)\,dx$$

$$= \int_0^{1.5} \left(30x+20x^2\right)dx+\int_0^{2} 90\,dx+\int_0^{2} (90+30x)\,dx$$

$$= \left[\frac{30x^2}{2}+\frac{20x^3}{3}\right]_0^{1.5}+[90x]_0^2+\left[90x+\frac{30x^2}{2}\right]_0^2$$

$$= 56.25+180+240$$

$$= 476.25$$

$$\Delta_{EV} = \frac{476.25}{EI}$$

Horizontal Deflection at 'E' = Δ_{EH}

$$EI\Delta_{EH} = \int Mm_2\, dx$$

$$= 0+0+\int_0^2 30x\,dx+\int_0^2 (30+10x)(x+2)\,dx$$

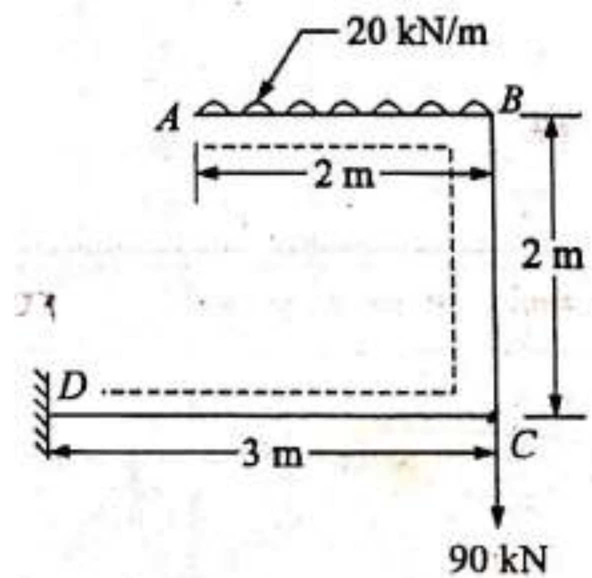
$$= \left[15x^2\right]_0^2+\int_0^2 \left(10x^2+50x+60\right)dx$$

$$= 60+\left[\frac{10x^3}{3}+50\times\frac{x^2}{2}+60x\right]_0^2$$

$$= 306.67$$

$$\Delta_{EH} = \frac{306.67}{EI}$$

Q4. Determine the vertical deflections at A and C in the frame shown in Figure by unit load method. Take E = 200 GPa, I = 150×10^4 mm^4.



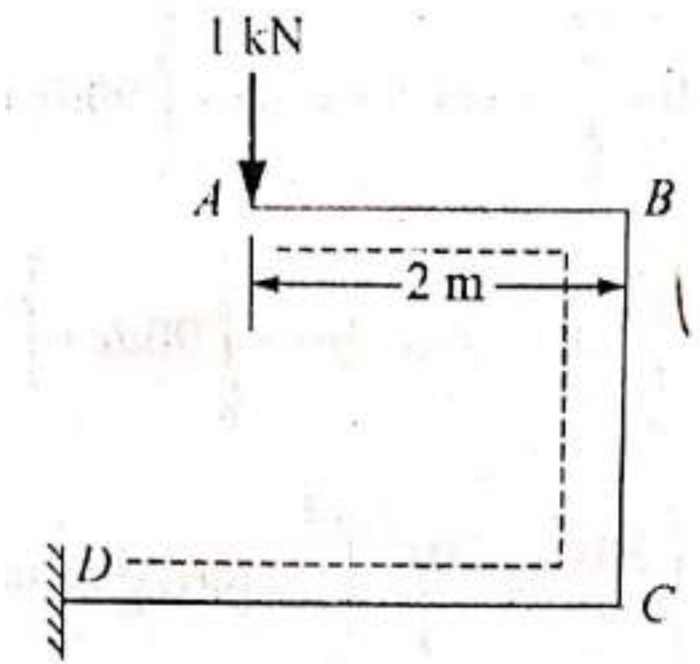


Figure 1: Frame with unit vertical lod at ‘*A*’

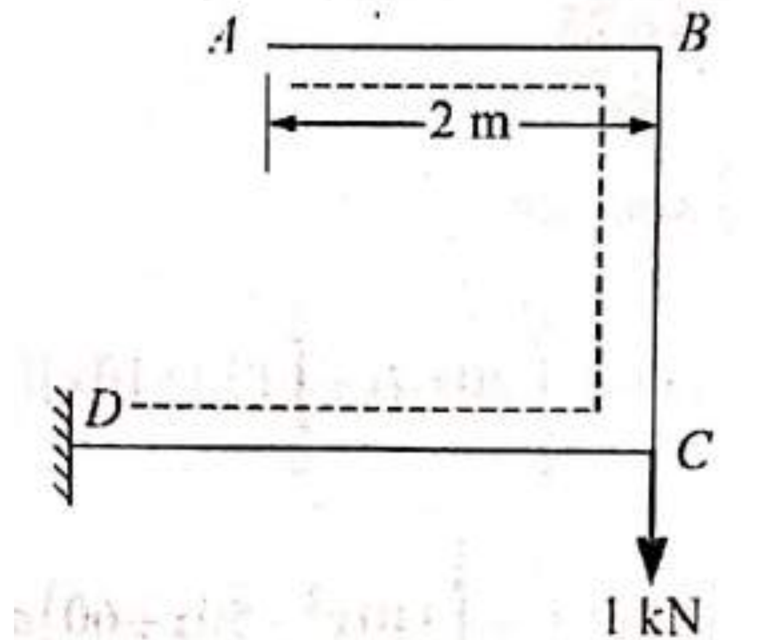


Figure 2: Frame with unit vertical load at ‘*C*’

- The bending moment expressions for 'M' due to given load, m_1 due to unit vertical load at A and m_2 due to unit vertical load at C are Tabulated below.

Portion	*AB*	*BC*	*CD*
Origin	A	B	C
Limit	0 – 2	0 – 2	0 – 3
M	$10x^2$	40	$40 - 130x$
m_1	x	2	$2 - x$
m_2	0	0	$-x$
Flexural Rigidity	EI	EI	EI

Vertical deflection at A = Δ_A

$$EI\Delta_A = \int_0^2 10x^2 \cdot x\,dx + \int_0^2 80\,dx + \int_0^3 (40 - 130x)(2 - x)\,dx$$

$$= \left[\frac{10x^4}{4}\right]_0^2 + [80x]_0^2 + \int_0^3 \left(80 - 300x + 130x^2\right)dx$$

$$= \frac{10(2^4)}{4} + 80(2) + \left[80x - 300\frac{x^2}{2} + \frac{130x^3}{3}\right]_0^3$$

$$= 260$$

$$E = 240 \text{ GPa} = 240 \times 10^9 \text{ N/m}^2$$

$$I = 150 \times 10^4 \text{ mm}^4 = 150 \times 10^4 \times 10^{-12} \text{ m}^4$$

$$= 150 \times 10^{-8} \text{ m}^4$$

$$\therefore \Delta = \frac{260}{240 \times 10^9 \times 150 \times 10^{-8}} = 7.222 \times 10^{-4} \text{ m}$$

$$= 0.722 \text{ mm}$$

Vertical Deflection at C = Δ_c

$$EI\Delta_c = \int Mm_2\,dx$$

$$= 0+0+\int_0^3 (40-130x)(-x)\,dx$$

$$= \int_0^3 \left(-40x+130x^2\right)dx$$

$$= \left[-20x^2+130\frac{x^3}{3}\right]_0^3$$

$$= 990$$

$$\Delta_C = \frac{990}{240\times10^9\times150\times10^{-8}} = 2.75\times10^{-3}\ \text{m}$$

$$= 2.75\ \text{mm}$$

Thanks